This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

MATHEMATICAL QUESTIONS,

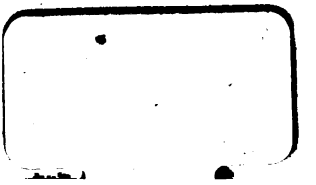
WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXXI.

600030305H



MATHEMATICAL QUESTIONS

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF
THE GENERAL MEDICAL COUNCIL.

VOL. XXXI.

FROM JANUARY TO JUNE, 1879.

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

1879.

18755. e. 2.

N.B.—Of this series thirty-one volumes have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, on all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

LIST OF CONTRIBUTORS.

ALDIS, J. S., M.A.; H.M. Inspector of Schools.
ALLEN, A. J. C., B.A.; Scholar of St. Peter's College, Cambridge.
ALLMAN, GEO. JOHNSTONE, LL.D.; Prof. of Mathematics in Queen's Univ., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor, Pesaro.
BALL, ROBT. STAWELL, LL.D., F.R.S.; Professor of Astronomy in the University of Dublin.
BATTAGLINI, GIUSEPPE; Professore di Matematiche nell' Università di Roma.
BELTRAMI, Professor; University of Pisa.
BERG, F. J. VAN DEN; Professor of Mathematics in the Polytechnic School, Delft.
BESANT, W. H., M.A.; Cambridge.
BIRCH, Rev. J. G., M.A.; Birmingham.
BLACKWOOD, ELIZABETH; Boulogne.
BLISSARD, Rev. J., B.A.; Hampstead Norris.
BORCHARDT, Dr. C. W.; Victoria Strasse, Berlin.
BOSANQUET, R. H. M., M.A.; Fellow of St. John's College, Oxford.
BOURNE, C. W., M.A.; Head Master of Bedford County School.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, COLIN, Professor in the Andersonian University, Glasgow.
BUCHHEIM, ARTHUR; New College, Oxford.
BURNSIDE, W. S., M.A.; Prof. of Mathematics in the University of Dublin.
CAMPBELL, Capt. FRED.; Notting Hill, London.
CAPEL, H. N., LL.B.; Bedford Square.
CARR, G. S.; Caius College, Cambridge.
CASEY, JOHN, LL.D., F.R.S.; Prof. of Higher Mathematics in the Catholic Univ. of Ireland.
CAVALLIN, C. B. S.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge; Member of the Institute of France, &c.
CHADWICK, W.; Fellow of Christ Ch., Oxford.
CHAKRAVARTI, BYOMAKESA; Calcutta.
CHASE, PLINY EARLE, LL.D.; Prof. of Philosophy in Haverford College.
CLARKE, Colonel A. R., C.B., F.R.S.; Director of the Ordnance Survey, Southampton.
COCHEZ, Professor; Paris.
COCKLE, Hon. Sir JAMES, Knt., M.A., F.R.S.; Chief Justice of Queensland; 5, Ladbroke Square, Notting Hill, London.
COHEN, ARTHUR, M.A., Q.C.; London.
COLSON, C. G., M.A.; University of St. Andrew's.
CONSTABLE, S.; Grammar School, Drogheda.
COTTERILL, J. H., M.A.; Royal School of Naval Architecture, South Kensington.
COTTERILL, THOS., M.A.; London, late Fellow of St. John's College, Cambridge.
CREMONA, LUIGI; Direttore della Scuola degli Ingegneri, S. Pietro in Vincoli, Rome.
CROFTON, M. W., B.A., F.R.S.; Professor of Mathematics and Mechanics in the Royal Military Academy, Woolwich.
CULVERWELL, E. P., B.A.; Sch. of Trin. Coll., Dubl.
DARBOUX, Professor; Paris.
D'ARCY, C. F., M.A.; Dublin.
DAVIS, R. F., B.A.; Wandsworth Common.
DAVIS, WILLIAM BARRETT, B.A. London.
DAY, Rev. H. G., M.A.; Riverside, Sevenoaks.
DICK, G. R., M. A.; Fellow of Caius Coll., Camb.
DOBSON, T., B.A.; Head Master of Hexham Grammar School.
DRACH, S. M.; Barnsbury Street, London.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême
DYER, J. M., B.A.; Cheltenham College.
EASTERBY, W., B.A.; Grammar School, St. Asaph.
EASTON, BELLE; Lockport, New York.
EDMUNDSON, GEORGE; Brasenose Coll., Oxford.
EDWARDES, DAVID; Stone Grove, Edgware.
ELLIOTT, E. B., M.A.; Fellow of Qu. Coll., Oxon.
ELLIS, ALEXANDER J., F.R.S.; Kensington.
ESCOTT, ALBERT, M.A.; Head Master of the Royal Hospital School, Greenwich.
EVANS, Prof. A. B., M.A.; Lockport, New York.
EVERETT, J. D., D.C.L.; Professor of Natural Philosophy in the Queen's University, Belfast.
FICKLIN, JOSEPH; Professor of Mathematics and Astronomy in the University of Missouri.
FORDE, S., M.A.; Oxford.
FORTEY, H., M.A.; Bellary, Madras Presidency.
FRY, Colonel JOHN H.; New York.
FUORTES, E.; University of Naples.
GALBRAITH, Rev. J., M.A.; Fell. Trin. Coll., Dublin.
GALTON, FRANCIS, M.A., F.R.G.S.; London.
GALLATLY, W., B.A.; Highgate.
GARDINER, MARTIN; late Professor of Mathematics in St. John's College, Sydney.
GENESE, R. W., M.A.; Sc. of St. John's Coll., Camb.
GERRANS, H. T., B.A.; Christ Church, Oxford.
GLAISHER, J. W. L., M.A., F.R.S.; Fellow of Trinity College, Cambridge.
GODWARD, WILLIAM; Wandsworth.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITH, W.; Superintendent of Public Schools, New London, Ohio, United States.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
HALL, Professor ASAPH, M.A.; Naval Observatory, Washington.
HAMMOND, J., M.A.; King Edward's Sch., Bath.
HARKEMA, C.; University of St. Petersburgh.
HARLEY, Rev. ROBERT, F.R.S.; Vice-Master of Mill Hill Grammar School.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HART, Dr. DAVID S.; Stonington, Connecticut.
HART, H.; R.M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dub.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, GEO.; Highfield, Weston-super-Mare.
HERBERT, A., M.A.; King Alfred's Sch., Wantage.
HERMAN, R. A., B.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HILL, Rev. E., M.A.; St. John's College, Camb.
HINTON, C. H.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; Director of Studies in the Royal Naval College, Greenwich.
HOPKINS, Rev. G. H., M.A.; Stratton.
HOPKINSON, J., D.Sc., B.A.; Holland Rd., London.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Fellow of St. John's College, Cambridge.
INGLEBY, C. M., M.A., LL.D., London.
JELLY, J. O., B.A.; Magdalen College, Oxford.
JENKINS, MORGAN, M.A.; London.
JENKINS, J. S.; Merton College, Oxford.
JOHNSON, J. M., B.A.; Radley College, Abingdon.
JOHNSON, W. W.; Annapolis, Maryland.
JOHNSTON, SWIFT; Trin. Coll., Dublin.
JONES, L. W., B.A.; Merton College, Oxford.
KEALY, J. A., M.A.; Wilmington, Delaware.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio, U.S.
KNOWLES, R., L.C.P.; Pentonville.
LADD, CHRISTINE; Professor of Natural Sciences and Mathematics, Union Springs, New York.
LAVERTY, W. H., M.A.; Public Examiner in the University of Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Cam.

LEIDHOLD, R., M.A.; Finsbury Park.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LEUDESDORF, C., M.A.; Fellow of Pembroke College, Oxford.
LONG, W. S. F.; St. John's College, Cambridge.
LOWRY, W. H., M.A.; Blackrock, Dublin.
MACDONALD, W. J., M.A.; Merch. Bank, Edin.
MACFARLANE, ALEX., D.Sc., F.R.S.E.; Gladstone Terrace, Edinburgh.
MCADAM, D. S.; Washington, Pennsylvania.
MCALISTER, DONALD, B.A., D.Sc.; St. Bartholomew's College, London.
MCCAY, W. S., M.A.; Fellow and Tutor of Trinity College, Dublin.
MCCOLL, HUGH; 73, Rue Sibliquin, Boulogne.
MCDOWELL, J., M.A.; Pembroke Coll., Camb.
MCLEOD, J., M.A.; R.M. Academy, Woolwich.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MALET, J. C., M.A.; Trinity College, Dublin.
MANNHEIM, M.; Professeur à l'École Polytechnique, Paris.
MARTIN, ARTEMAS, M.A.; Editor and Publisher of the *Mathematical Visitor*, Erie, Pa.
MARTIN, Rev. H., D.D., M.A.; Examiner in Mathematics in the University of Edinburgh.
MATZ, Prof., M.A.; King's Mountain, Carolina, U.S.
MERRIFIELD, C. W., F.R.S.; Brook Green.
MERRIFIELD, J., LL.D., F.R.A.S.; Plymouth.
MERRICK, THOS.; Kensington Square, London.
MERRIMAN, MANSFIELD, M.A.; Yale College.
MILLER, W. J. C., B.A.; 55, Netherwood Road, West Kensington Park, London, W.
MINCHIN, G. M., M.A.; Prof. in Cooper's Hill Coll.
MITCHESON, T., B.A., L.C.P.; City of London Sch.
MONCK, HENRY STANLEY, M.A.; Prof. of Moral Philosophy in the University of Dublin.
MONCOURT, Professor; Paris.
MONRO, J., M.A.; Hadley, Barnet.
MOON, ROBERT, M.A.; Ex-Fell. Qu. Coll., Cam.
MOORE, E. H., M.A.; Dublin.
MOREL, Professor; Paris.
MORLEY, THOS., L.C.P.; Bromley, Kent.
MORLEY, F., B.A.; Woodbridge, Suffolk.
MOULTON, J. F., M.A.; Fell. of Ch. Coll., Camb.
MURPHY, HUGH; Head Master of the Incorporated Society's School, Dublin.
NARENDRA LAL DEY; Presidency Coll., Calcutta.
NASH, A. M., B.A.; Professor of Nat. Phil. and Astronomy, Presidency College, Calcutta.
NELSON, R. J., M.A.; Naval School, London.
O'REGAN, JOHN; New Street, Limerick.
ORCHARD, H. L., B.A., L.C.P.; Hampstead.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PILLAI, C. K.; Trichy, Madras.
PIRIE, A., M.A.; University of St. Andrews.
POLIGNAC, Prince CAMILLE DE; Paris.
POLLEXFEN, H., B.A.; Windermere College.
PRUDDEN, FRANCES E.; Lockport, New York.
PURSER, F., M.A.; Rathmines Castle, Dublin.
RAWSON, ROBERT; Havant, Hants.
RENSHAW, S. A.; Nottingham.
RILEY, R. E., B.A., Bournemouth.
RIPPIN, CHARLES R., M.A.; Woolwich Common.
ROBERTS, R. A., B.A.; Schol. of Trin. Coll., Dublin.
ROBERTS, SAMUEL, M.A.; Tufnell Pk., London.
ROBERTS, Rev. W., M.A.; Senior Fellow of Trinity College, Dublin.
ROBERTS, W. R., M.A., Ex-Sch. of Trin. Coll., Dub.
ROSENTHAL, L. H.; Scholar of Trin. Coll., Dublin.
ROYDS, J., L.C.P.; Kiveton Park, Sheffield.
RÜCKER, A. W., B.A.; Professor of Mathematics in the Yorkshire College of Science, Leeds.
RUGGERO, SIMONELLI; Università di Roma.
RUTTER, EDWARD; Sunderland.
SALMON, Rev. G., D.D., F.R.S.; Regius Professor of Divinity in the University of Dublin.
SANDERS, J. B.; Bloomington, Indiana.
SANDERSON, Rev. T. J., M.A.; Royston, Cambs.
SARKAR, NILKANTHA, B.A.; Calcutta.
SAVAGE, THOMAS, M.A.; Fell. of Pem. Coll., Cam.
SCHEFFER, Professor; Des Moines, Iowa, U.S.
SCOTT, A. W., M.A.; St. David's Coll., Lampeter.
SCOTT, JOSIAH, Judge of the Ohio Supreme Court, Bucyrus, United States.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Cam.
SEITZ, E. B.; Greenville, Ohio, United States.
SERRET, Professor; Paris.
SHARP, W. J. C., M.A.; Grosvenor Square.
SHARPE, J. W., M.A.; The Charterhouse.
SHARPE, Rev. H. T., M.A.; Cherry Marham.
SHEPHERD, A. J. P.; Queen's College, Oxford.
SIDES, J. J.; Rue des Vieillards, Boulogne.
SIVERLY, WALTER; Oil City, Pennsylvania.
SMITH, C., M.A.; Sidney Sussex Coll., Camb.
SPOTTISWOODE, WILLIAM, M.A.; President of Royal Society, Grosvenor Place, London.
STARENOW, H., M.A.; New York.
STEGGALL, J. E. A., B.A.; Clifton.
STEIN, A.; Venice.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge.
SWIFT, C. A., B.A.; Grammar Sch., Weybridge.
SYMONS, E. W.; University Coll., Oxford.
SYLVESTER, J. J., LL.D., F.R.S.; Professor of Mathematics in Johns Hopkins University, Member of the Institute of France, &c.
TAIT, P. G., M.A.; Professor of Natural Philosophy in the University of Edinburgh.
TANNER, H. W. L., M.A.; London.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dub.
TAYLOR, Rev. C., M.A.; Fell. St. John's Coll., Cam.
TAYLOR, H. M., M.A.; Fellow and Assistant Tutor of Trinity College, Cambridge.
TEBAY, SEPTIMUS, B.A.; Farnworth, Bolton.
TERRY, T. A., M.A.; Magdalen Coll., Oxford.
THOMAS, Rev. DAVID, M.A.; Garsington Rectory, Oxford.
THOMSON, F. D., M.A.; late Fellow of St. John's Coll., Camb., Brinkley Rectory, Newmarket.
TODHUNTER, ISAAC, F.R.S.; Cambridge.
TOMLINSON, H.; Christ Church, Oxford.
TORELLI, GABRIEL; University of Naples.
TORRY, Rev. A. F., M.A.; St. John's Coll., Cam.
TOWNSEND, Rev. R., M.A., F.R.S.; Professor of Nat. Phil. in the University of Dublin, &c.
TRAILL, ANTHONY, M.A., M.D.; Fellow and Tutor of Trinity College, Dublin.
TROWBRIDGE, DAVID; Waterburgh, New York.
TUCKER, R., M.A.; Mathematical Master in University College School, London.
TURRELL, I. H.; Cumminsville, Ohio.
TURRIFF, GEORGE, M.A.; Aberdeen.
VINCENZO, CECCHINI, University of Rome.
VINCENZO, JACOBINI, University of Rome.
VOSE, G. B.; Professor of Mechanics and Civil Engineering, Washington, United States.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, J. J., M.A.; Vice-Principal of University Hall, Gordon Square, London.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARD, ISABELLA M.; Capecure, Boulogne,
WARREN, R., M.A.; Trinity College, Dublin.
WATSON, STEPHEN; Haydonbridge.
WATSON, H. W., M.A.; Ex-Fell. Trin. Coll., Cam.
WERTSCH, Fr.; Weimar.
WHITE, J. R., B.A.; Worcester Coll., Oxford.
WHITE, Rev. J., M.A.; Cowley College, Oxford.
WHITWORTH, Rev. W. A., M.A.; Fellow of St. John's Coll., Camb.; Hammersmith.
WILKINS, W.; Scholar of Trin. Coll., Dublin.
WILKINSON, Rev. M. M. U.; Norwich.
WILLIAMSON, B., M.A.; Fellow and Tutor of Trinity College, Dublin.
WILSON, J. M., M.A.; Clifton College.
WILSON, Rev. J., M.A.; Rector of Bannockburn Academy.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WILSON, Rev. R., D.D.; Chelsea.
WOLSTENHOLME, Rev. J., M.A.; Professor of Mathematics in Cooper's Hill College.
WOOLHOUSE, W. S. B., F.R.A.S., &c.; London.
WRIGHT, Dr. S. H., M.A.; Penn Yan, New York.
WRIGHT, Rev. W. J., Ph.D.; Pennsylvania.
YOUNG, JOHN, B.A.; Academy, Londonderry.

CONTENTS.

Mathematical Papers, &c.

No.		Page
153.	Note on the Solution of Question 1819. By E. B. Seitz.	27
154.	Note on Landen's Theorem. By Christine Ladd.	39
155.	Note on Mr. Stabenow's Solution of Question 5832. By J. Hammond, M.A.....	45
156.	First Note on Some Questions in Probability. By the Editor...	100
157.	Second Note on Some Questions in Probability. By the Editor.	101
158.	Note on the Solution of Question 1401. By the Editor.....	112

Solved Questions.

1401. (The Editor).—Eliminate x from the equations

$$x^5 + px^3 + qx + r = 0, \quad x^4 + dx^3 + cx^2 + bx + a = y,$$

and exhibit the result in the form

$$y^5 + Ay^4 + By^3 + Cy^2 + Dy + E = 0,$$

giving the values of A, B, C, D, E in terms of p, q, r, a, b, c.... 111

2495. (W. S. B. Woolhouse, F.R.A.S.)—Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude, such that, if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum..... 61

3560. (Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelope will be $\frac{1}{4}$PQ (cot θ + cot ϕ), θ and ϕ being the angles at which PQ cuts the given curve..... 22

3633. (Rev. G. H. Hopkins, M.A.)—From any point on the surface of a circular cylinder planes are drawn; find the equation to the surface upon which the foci of all the elliptic sections are placed; and prove that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the logocyclic curve. 35

3743. (T. Cotterill, M.A.)—If a line, length k, rest upon the curve $(xy - ak)^2 = (x + b)^2 (k^2 - x^2)$ and the line $x = 0$, it envelopes a circle and a parallel to the four-cusped hypocycloid..... 24

4323. (Professor Wolstenholme.)—If two points P, Q describe the same circle in the same sense with velocities always in the

No. Page

ratio $m : n$ (m, n being whole numbers prime to each other and $n > m$), the envelope of PQ is an epicycloid whose vertices lie on the given circle and cusps on a concentric circle, whose radius is $\frac{n-m}{n+m}$ times that of the given circle. The class is $n+m$ and the degree $2n$, and the points of contact of tangents drawn from any point on the given circle will be corners of two regular polygons of m sides and n sides respectively. The circular points at infinity are multiple points of the nth degree, of form similar to the origin in the curve $Z^{m+n} \propto$, X^n, where Z is the line at infinity and X the straight line joining it to the centre of the given circle.

If the points move in opposite senses, the envelope is an hypocycloid of degree $2n$ and class $m+n$, the other properties being the same, but for $\frac{n-m}{n+m}$ we write $\frac{n+m}{n-m}$, and for the form of the circular points at infinity, $Z^{n-m} \propto X^m$ or $\propto Y^m$. 25

5033. (S. Tebay, B.A.)—The roots of a cubic equation are x_1, x_2, x_3. Given $x_1 + x_2 + x_3 = a$, and $x_1 > x_2 > x_3$; show that the mean form of the equation is

$$x^3 - ax^2 + \tfrac{1}{4}a^2 - \tfrac{1}{30}a^3 = 0 \text{ ..... } 78$$

5062. (W. H. H. Hudson, M.A.)—Trace the curves

$$2\{x^2-(y-a)^2\}\{4x^2-(y-2a)^2\} + 9y^2\{x^2+a(y-a)\} = 0 \text{ ...(1)},$$

$$a(a^2-\beta\gamma)(\beta+\gamma) = \beta^2\gamma^2 \text{ .... (2).} \quad 30$$

5141. (J. J. Walker, M.A.)—Show that the axis of inflexion of the cubic $ax^3 + by^3 + 3fy^2z + 3hx^2y + 3jzx^2 + 3kxy^2 = 0$ is $(af+jk)x + (bj+fh)y + 4fjz = 0$. 25

5168. (Artemas Martin, M.A.)—Through a point taken at random in the surface of a circle, two chords are drawn, one at random, and the other at right angles to the radius through that point; show that the average area of the quadrilateral formed by joining the ends of the chords is one-fourth of the area of the circle. 75

5275. (Professor Wolstenholme, M.A.)—Prove that (1) the locus of the points of intersection of circles of curvature of a parabola drawn at the ends of a focal chord is a bicircular quartic having an acnode and two double tangents; (2) the equation referred to the acnode as origin is $(x^2+y^2-2ax)^2 = 4a^2(3y^2+7x^2)$, that of the parabola being $y^2 = 4a(x+3a)$; (3) the bitangents are inclined at 30° to the axis, and the equation referred to their point of intersection is

$$(r^2 - 14ar\cos\theta + 36a^2)^2 = 4a^2r^2(4\cos^2\theta - 3),$$

the curve being its own inverse with respect to this point. [It would be difficult to show the double tangents in a figure, since for all four values of r to be real we must have $\cos\theta > \frac{1}{2}\sqrt{3}$ and $< \frac{1}{10}\frac{3}{\sqrt{3}}$, and the limiting values differ by about 5′ 50″.] (4) The curve osculates the parabola in the two points whose distance from the focus is equal to the latus rectum. (5) The equation of the curve referred to the bitangents is

$$[x^2+y^2+xy-7b(x+y)+12b^2]^2 = 4b^2xy.$$

(6) Two circles of curvature touch each other and also this

No. Page

locus at a point whose distances from the node and the centre of inversion are as 5 : 8, being $\frac{10a}{\sqrt{11}}$ and $\frac{16a}{\sqrt{11}}$. 27

5380. (W. Gallatly, B.A.)—If a circle A touches internally another circle B at P, and a tangent to A at the point Q intersect B in R_1, R_2, prove that $\angle R_1PQ = \angle R_2PQ$. 65

5387. (Professor Cayley, F.R.S.)—Show that a cubic surface has at most 4 conical points, and a quartic surface at most 16 conical points. 38

5390. (Professor Lloyd Tanner, M.A.) — When an equation $(F\ s, p, q, z, x, y) = 0$ has two independent first integrals, each involving an arbitrary function, it can be reduced to the form $\frac{d^2\zeta}{dx\,dy} = 0$, where ζ is a function of z, x, y. 77

5493. (Professor Sylvester, F.R.S.)—If χ_q represent in general the number of linearly independent covariants of the degree q in the variables, and of a given order j in the constants belonging to a binary quantic of the degree i, prove that

$$\chi_0 + 2\chi_1 + 3\chi_2 + 4\chi_3 + 5\chi_4 + \ldots\ldots = \frac{\Pi(i+j)}{\Pi i \,.\, \Pi j}. \quad \ldots\ldots 87$$

5527. (W. S. B. Woolhouse, F.R.A.S., &c.)—A given triangle may be orthogonally projected from an equilateral triangle; or it may be orthogonally projected into an equilateral triangle. Determine, by an easy geometrical construction, the magnitudes of these equilateral triangles..... 18

5536. (W. H. H. Hudson, M.A.)—Trace the curve

$$(x^2 + y^2 + 8c^2)^2(x^2 + y^2 - c^2) = 108c^2x^2y^2,$$

and prove that its evolute is similar to itself. 24

5545. (J. C. Malet, M.A.)—Prove that the locus of the centre of a circle which cuts orthogonally a given circle, and touches a given conic, is a curve of the sixth order, which has six cusps that lie all on another conic; and hence prove that the six cusps of the first negative pedal of a central conic with respect to any point all lie on another conic. 93

5578. (J. Hammond, M.A.) — Show (1) that the sum to $2m+1$ terms of the series

$$\frac{1}{n} - \frac{2m+n}{n(n+1)} + \frac{(2m+n)(2m+n-1)}{n(n+1)(n+2)} - \frac{(2m+n)(2m+n-1)(2m+n-2)}{n(n+1)(n+2)(n+3)} + \&c.$$

is $(n+m)^{-1}$; and (2) find the sum to infinity when both m and n are fractional, as also $2m+n$ 29

5612. (S. Roberts, M.A.)—Prove that (1) the locus of points such that the sum of the squares of their normal distances from a given conicoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ is constant, is a coaxial and concentric conicoid; (2) if the constant sum is $2(a^2 + b^2 + c^2)$, the locus becomes a cone; and (3) for a particular value of the constant sum it is identical with the given conicoid, if this is one of a certain family of real similar ellipsoids. 79

No. Page

5627. (Professor Townsend, F.R.S.)—A cylinder of revolution of uniform density being supposed to float, with axis vertical, in a gravitating fluid of greater density; show, by any method, that the coaxal spheroid of revolution, the square of whose equatoreal is double of that of its polar radius, will intersect its surface in the limiting circle of floatation consistent with the stability of its equilibrium under the action of gravity 74

5634. (Christine Ladd.)—A secant, bisected in E, meets the successive sides of a quadrilateral inscribed in a circle in the points P, I, Q, H respectively; show that, if EI = EH, EP = EQ..... 92

5664. (Professor Tait, F.R.S.)—Show that the greatest amount of mechanical effect which can be obtained from a system of equal and similar masses (whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures. 93

5674. (E. B. Seitz, M.A.)—Two points are taken at random in the surface of a square; show that the chance that the square, whose diagonal is the line joining the points, lies wholly within the square, is $\frac{1}{2}$. 84

5680. (W. J. C. Sharpe, B.A.)—Find (1) the invariant conditions that two conics should have double contact; and (2) the equation to the common tangents in terms of the quantics and their invariants. 86

5690. (Professor Townsend, F.R.S.) — Two planets, moving in different planes and describing any orbits round the Sun as common centre of attraction, being supposed to appear stationary in the heavens as observed from each other: if p and p' be their perpendicular distances from the line of nodes of their orbits, and h and h' the synchronous areas of their orbital motions, shew that $p : p' = h : h'$. 17

5721. (Elizabeth Blackwood.)—A line is drawn at random through a random point within a circle, and another line is drawn at random through the centre; find the chance that these lines will intersect within the circle..... 22

5728. (By F. P. Matz, M.A.)—An equilateral spherical triangle is (1) inscribed in, (2) circumscribed about, a circle of radius ϕ; show that the cosine of an angle of the triangle is

$$\frac{3\cos^2\phi - 1}{3\cos^2\phi + 1} \text{ in (1), and } \tfrac{3}{2}\cos^2\phi - 1 \text{ in (2).} \ldots\ldots 76$$

5732. (W. Gallatly, M.A.) — Find, without differentiation, the envelope of $x\cos^3\theta + y\sin^3\theta = a$..... 83

5742. (Frederick Purser, M.A.)—A conic U touches a line L and a conic V, and has double contact with another conic W; show that the chord of contact of U and W envelopes a binodal quartic of which the points of intersection of L and W are the nodes. 97

5746. (C. Leudesdorf, M.A.)—The circle of curvature at any point P of the curve $ay^2 = x^3$ cuts the curve in three other points: prove that the locus of the centroid of these three points, as p moves along the curve, is $3ay^2 + (3x + 2a)(7x + 2a)^2 = 0$. 23

No. Page

5748. (J. J. Walker, M.A.)—If uyz satisfy $\alpha x+\beta y+\gamma z=0$, and $u=0$, u being a ternary form of the order p, show how to transform

$$\beta\gamma^2 y\left\{\frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2-2\frac{d^2u}{dx\,dy}\frac{du}{dx}\frac{dx}{dy}\right\}$$
$$+\beta\gamma z\left\{\frac{d^2u}{dx^2}\left(\frac{du}{dz}\right)^2+\frac{d^2u}{dz^2}\left(\frac{du}{dx}\right)-2\frac{d^2u}{dz\,dx}\frac{du}{dz}\frac{du}{dx}\right\}$$

into the symmetrical form

$$\alpha\beta\gamma\left\{(p-1)\frac{du}{dx}\frac{du}{dy}\frac{du}{dz}+x\frac{d^2u}{dy\,dz}\left(\frac{du}{dx}\right)^2+y\frac{d^2u}{dz\,dx}\left(\frac{du}{dy}\right)^2+z\frac{d^2u}{dx\,dy}\left(\frac{du}{dz}\right)^2\right\}.$$

..... 34

5766. (Professor Minchin, M.A.)—The vector to any point on a plane curve being ρ, and the successive s-fluxes of ρ being denoted by ρ', ρ'', ρ''', &c., give a simple construction for the direction of the vector ρ'''. 17

5768. (Professor Dupain.)—Déterminer sur un diamètre AB d'une sphère de rayon R, un point tel que si l'on mène par ce point un plan perpendiculaire à ce diamètre, la surface de la zône sphérique limitée par le plan et contenant le point A, soit équivalente à la surface latérale du cône qui a pour base le cercle d'intersection de la sphère et du plan et pour sommet le point B. Cela étant, calculer le rapport du volume de ce cône au volume de la sphère. 59

5772. (C. Leudesdorf, M.A.)—If the areas of the pedals of any given closed curve with regard to four points A, B, C, P in its plane, taken successively as pedal origin, be denoted by (A), (B), (C), (P) respectively, prove that

$$(P)=x(A)+y(B)+z(C)+\tfrac{1}{2}\pi t^2,$$

where x, y, z are the areal coordinates of P referred to the triangle ABC, and t is the length of the tangent from P to the circle around ABC... 75

5783. (S. Tebay, B.A.)—If m cards, consisting of s suits of a_1, a_2, ... a_s cards respectively, be shuffled, find the probability that each suit consists of p partitions. 50

5786. (A. Martin, M.A.)—A square hole is cut through the centre of a sphere of radius r; show that the average volume removed is $\frac{11}{12}(23\sqrt{2}-28)\pi r^3$. 65

5790. (E. W. Symons.)—Prove that the equation of the tangent to an involute of $u=0$, at the point corresponding to (xy), is

$$(X-x)\frac{du}{dy}-(Y-y)\frac{du}{dx}=s\left(\frac{du^2}{dx^2}+\frac{du^2}{dy^2}\right)^{\frac{1}{2}},$$

s being the length of the arc of u measured from a certain fixed point on the curve to (x, y)..... 92

5792. (Rev. Dr. Haughton, F.R.S.)—Show that, within the limits of Newton's construction for the disturbing force, the *exact equation* of the surface of an equatorial canal of constant width and depth, attracted by the moon always in the equator, is $\rho=-a\cos 2\phi+b$; also trace the form of the curve for the following particular cases:—(1) $b=0$; (2) $b<a$; (3) $b=a$; (4) $b>a$, but $<5a$; (5) $b=$ or $>5a$. 31

No. Page

5793. (Professor Sylvester, F.R.S.)—

If $u_i = \left\{\left(\frac{d}{dx}\right)^i f(x)\right\}^2 - \frac{A+Bi-B}{A+Bi}\left\{\left(\frac{d}{dx}\right)^{i-1} f(x) \cdot \left(\frac{d}{dx}\right)^{i+1} f(x)\right\}$,

prove that, when $u_i = 0$, $u_{i+1} = 0$, ..., $u_{i+w} = 0$,

$$\frac{\left(\frac{d}{dx}\right)^w u_i}{u_{i+w}} = \frac{A+B(i+w)}{A+Bi} \cdot \frac{\left(\frac{d}{dx}\right)^i f(x)}{\left(\frac{d}{dx}\right)^{i+w} f(x)}. \quad \ldots\ldots\ldots \quad 36$$

5798. (Professor Cochez.)—Construire un triangle rectangle, connaissant les sommes de chacun des côtés avec la perpendiculaire abaissée du sommet de l'angle droit sur l'hypotenuse..... 56

5802. (T. R. Terry, M.A.)—If O be a fixed point in a rigid body, and the direction cosines of the lines OA, OB, OC be represented by (l, m, n), (l', m', n'), (λ, μ, ν), respectively; show that a rotation of the body round the axis OA through an angle 2θ, followed by a rotation round the axis OB through an angle $2\theta'$, is equivalent to a rotation round the axis OC through an angle 2ϕ, where

$$\frac{\cos\phi}{\cos\theta'\cos\theta - \sin\theta'\sin\theta\,(ll'+mm'+nn')} = \frac{\lambda\sin\phi}{X} = \frac{\mu\sin\phi}{Y} = \frac{\nu\sin\phi}{Z},$$

where

$X = l\cos\theta'\sin\theta + l'\cos\theta\sin\theta' + \sin\theta'\sin\theta\,(m'n - mn')$, Y, Z = &c. 58

5809. (The Editor.)—If O be any point within a triangle ABC, prove that $OA^n \sin 2A + OB^n \sin 2B + OC^n \sin 2C$ is least when O is the centre of the circumscribed circle. 80

5813. (W. R. Roberts, M.A.)—From any point on an inflexional tangent to a cubic can be drawn four tangents to the curve. Show that there are six points, two on each inflexional tangent lying on a conic, from which the tangents form an harmonic pencil. 98

5814. (H. L. Orchard, L.C.P.)—Two spherical soap-bubbles are blown, the one from water, the other from a mixture of water and alcohol. If the unit superficial tensions per linear inch be 1 grain and $\frac{7}{11}$ grain respectively, and the radii be $\frac{7}{8}$ inch and $1\frac{1}{2}$ inch respectively; compare, in the two case, the excess of the total internal pressure over the external pressure..... 96

5818. (F. C. Wace, M.A.)—If P_1, P_2, P_3, be the three forces which, acting along the surface of the plane at inclinations of β_1, β_2, β_3 to the line of greatest slope, will support the same weight, prove that

$$P_1\cos\beta_1(P_2^2 - P_3^2) + P_2\cos\beta_2(P_3^2 - P_1^2) + P_3\cos\beta_3(P_1^2 - P_2^2) = 0. \quad 23$$

5819. (Prince Camille de Polignac.)—On each side of a triangle as segment, an internal arc of a circle is drawn containing an angle of 120°; (1) prove that the coordinates of the common point of intersection of the three circles referred to the given triangle are proportional to

$$\frac{1}{\sin A + \sqrt{3}\cos A}, \quad \frac{1}{\sin B + \sqrt{3}\cos B}, \quad \frac{1}{\sin C + \sqrt{3}\cos C};$$

and (2) find a characteristic feature of the point in relation to the triangle. 40

No. Page

5823. (Professor Townsend, F.R.S.)—A uniform cylindrical elastic rod, supported at both extremities and at both points of trisection by four rigid props in the same horizontal right line, being supposed in strained equilibrium, with slight deflection of form, under the action of gravity; required the distribution of its weight between the props, with the points of inflexion of its strained axis arising from their reactions. 46

5828. (Professor Darboux.)—On coupe une pyramide triangulaire SABC par un plan parallèle à la base; ce plan rencontre les arètes latérales SA, SB, SC en A′, B′, C′; on mène ensuite les plans CA′B′, AB′C′, BC′A′. Soit P leur point commun. Déterminer le lieu décrit par le point P lorsque le plan A′BC′ se déplace en demeurant parallèle à la base. 57

5829. (Professor Cochez.) — Trouver les conditions pour que l'équation du quatrième degré $Ax^4Cx^2Dx\mathrm{F} = 0$ ait trois racines égales. On trouve les 2 conditions

$$\left(\frac{2}{2}\right)^3\left(\frac{\mathrm{D}}{\mathrm{A}}\right)^2 = 12\frac{\mathrm{CF}}{\mathrm{A}^2} = -\frac{\mathrm{C}^3}{\mathrm{A}^3}. \quad \ldots\ldots 66$$

5831. (Professor Minchin, M.A.) — If S and E be a conic and its evolute, and if N be the conic whose intersections with S are the feet of the normals drawn to S from a point P: prove that (1) the perpendicular from P on its polar with respect to S is a tangent to N at P; (2) the centre of gravity of the feet of the four normals is the middle point of the line joining the centres of S and N; (3) the locus of points whence normals drawn to S form a harmonic pencil is $\mathrm{E} + 27a^2b^2c^4x^2y^2 = 0$; (4) E being $\equiv \mathrm{C}^3 + 27a^2b^2c^4x^2y^2$, if P be any point on the conic C, the locus of the centre of N is S. 54

5832. (The Editor.)—In Booth's *Geometrical Methods*, Vol. I., p. 114, it is shown that the sections of the surface of centres (S) by a principal plane of the ellipsoid are an ellipse and the evolute of an ellipse. Prove that the principal plane *intersects* on one sheet of the surface in the evolute section, and *touches* the other sheet in the elliptic section. And, generally, that the normals at two consecutive points of a line of curvature of the ellipsoid *intersect* on one sheet of the surface of centres, while their plane touches the other sheet. 44

5834. (J. J. Walker, M.A.)—Referring to Quest. 4372 (*Reprint*, Vol. XXIII., p. 30), show that, if OD, OE, OF, perpendicular respectively to OA, OB, OC, meet BC, CA, AB respectively in D, E, F (three collinear points, as Prof. Kelland has shown); then, if O is a point on the circumference passing through the middle points of BC, CA, AB, the transversal DEF passes through the centre of the circle circumscribing ABC. 35

5849. (Sir James Cockle, F.R.S.)—Find the general solution of the linear partial differential equation of the first order

$$\frac{n}{z}(z^m + \mathrm{A}z^r + \mathrm{B})(p+q) = \frac{n}{n-r}\left(\frac{d\mathrm{A}}{dx} + \frac{d\mathrm{A}}{dy}\right)z^r + \frac{d\mathrm{B}}{dx} + \frac{d\mathrm{B}}{dy} \ldots\ldots(1),$$

wherein n, m, and r are unequal, and A and B are free from z, and are not functions one of the other. 39

5851. (Professor Townsend, F.R.S.)—A material particle being supposed in equilibrium, in a free space, under the action, attrac-

No. Page

tive or repulsive, of a system of masses distributed in any manner in the space; show, generally, that its equilibrium, when not absolutely neutral, is always perfectly stable for attractive, and perfectly unstable for repulsive, action, or is, on the contrary, always intermediate, according as the law of the action is a direct or an inverse power of the distance. 59

5852. (Professor Minchin, M.A.)—Any rigid body is kinetically equivalent to three equal uniform spheres whose centres are the vertices of a maximum triangle inscribed in a certain ellipse lying in one of the principal planes at the centre of gravity of the body.

Prove this geometrically from the fact that, if two intersecting rectangular lines of constant lengths resolve in a fixed plane about their point of intersection, a point whose abscissa and ordinate are the projections of these lines on any fixed line, will trace out an ellipse. 37

5853. (Professor Crofton, F.R.S.)—Explain the following paradox:—One bag (A) contains 5 cheques, whereof 2 are known to be over £5 and 3 under. Another bag (B) contains 7 cheques, 4 over £5 and 3 under. Find the chance, if a cheque be drawn from A, that it exceeds in amount one drawn from B. There are 35 cases, all equally probable; 6 evidently favourable, 12 unfavourable, and 17 doubtful. In these 17 cases, as all we know is that the 2 cheques drawn are both over £5, or both under, it is an even chance that one exceeds the other. Hence the chance required, that the cheque from A exceeds that from B, is $\frac{6}{35} + \frac{1}{2}(\frac{17}{35}) = \frac{29}{70}$. But the chance that the cheque from A exceeds £5 is $\frac{2}{5} = \frac{28}{70}$, so that there is a *less* chance that the cheque drawn from A exceeds £5 than that it exceeds an (unknown) quantity which is likely to be greater than £5; the odds being 4 to 3. *A priori*, one would say it must be the other way. 54

5855. (Professor Darboux.)—Par le point de contact A de deux circonférences données, on mène deux cordes AB et AD qui soient dans un rapport donné; et des centres O, C on abaisse des perpendiculaires sur ces cordes. On demande le lieu géométrique du point de rencontre M de ces perpendiculaires. 53

5859. (J. J. Walker, M.A.)—If α, β, γ are any three vectors, show that $(T\alpha)^2 S\beta\gamma + S\gamma\alpha S\alpha\beta = S(V\alpha\beta \cdot V\alpha\gamma)$, and verify the result trigonometrically. 81

5861. (Hugh McColl, B.A.)—Required a solution of the following logical problem:—Suppose it to have been ascertained by observation (1) that, whenever the events A and B happen together, they are invariably followed by the event C, and also by either the event D or the event E; and (2) that, whenever the events D and E both happen, they have invariably been preceded by the event A, or else by both the events B and C. When may we conclude (from the occurrence or non-occurrence of the events A, B, C, D)—(1) that E will certainly happen; (2) that he will certainly *not* happen? 43

5862. (Edwyn Anthony, M.A.)—Prove that $n^{13} - n$ is always divisible by 2730, whatever whole number n may be. 56

No. Page

5864. (J. Young, B.A.)—Find in what proportion tea at 3*s.* per lb. is mixed with tea at 4*s.* per lb., when by reversing the proportion the quality of the mixture would be improved 10 per cent. 83

5866. (H. N. Capel, LL.B)—Given the two lines bisecting the acute angles of a right-angled triangle; determine the triangle..... 53

5867. (Rev. W. A. Whitworth, M.A.)—Construct 40 different right-angled triangles having the common hypothenuse 32045, and all their sides integers..... 67

5870. (J. L. Mackenzie, B.A.)—The evolute of a non-singular cubic has in general twenty-seven cusps. If P be a point on the cubic corresponding to any of these cusps, shew that the remaining points in which the circle of curvature at P cuts the cubic are collinear with the *second tangential* of P. 57

5872 & 5891. (E. B. Seitz and S. Roberts, M.A.)—A line is drawn at random across a circle, and a point is taken at random in each segment; show that (1) the average distance between the points is $\frac{45}{128}\pi r$, and (2) the average length of the chord is $\frac{75}{128}\pi r$..... 70

5875. (Rev. Dr. Haughton, F.R.S.)—Show that, within the limits of Newton's construction for the disturbing force, the *equilibrium tidal surface* (statical problem) of an equatorial canal of constant width and depth (without rotation), attracted by the moon always in the equator, is

$$(\log \rho) = \alpha \cos 2\phi + \beta. \quad \text{.....} \quad 52$$

5881. (Professor Darboux.)—Sur un cercle donné O, on prend à volonté un arc ANB, et, sur la corde AB de cet arc, on décrit une demicirconférence AMB, puis on fait tourner la figure autour du diamètre perpendiculaire à AB. On demande quelle doit être cette corde pour que la somme des surfaces décrites par les lignes AMB, ANB, soit maximum 90

5885. (The Editor.)—Determine a right-angled triangle, having given the two lines bisecting the acute angles, and terminated by the opposite sides; and show that, if the lengths of the bisectors be 40 and 50, the lengths of the sides will be 35·80738, 47·40728, 59·41143. 88

5887. (Edwyn Anthony, M.A.)—If O be the middle point of a focal chord PSQ of an ellipse, and SP equals its semi-major-axis; show that the ratio of OS to OP equals the square of the eccentricity. 57

5888. (R. Tucker, M.A.)—D, E, F are the points of contact of an inscribed circle with the sides. Circles are inscribed in AFE, BFD, CDE, and their centres are joined. The triangle A′B′C′ thus formed is treated in the same way as the triangle ABC, and so on: prove that the triangle $A_\infty B_\infty C_\infty$ is equilateral..... 99

5892. (Rev. W. A. Whitworth, M.A.)—If C_r^n denote the number of combinations of n things r together, prove that

$$\Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \Sigma_{\gamma=0}^{\gamma=c} \dots \Sigma_{\mu=0}^{\mu=m} \frac{C_\alpha^a C_\beta^b C_\gamma^c \dots C_\mu^m}{C_{\alpha+\beta+\gamma+\dots+\mu}^{a+b+c+\dots+m}} = 1 + a + b + c + \dots + m. \quad \text{.....} \quad 95$$

No. Page

5896. (E. W. Symons.)—Prove that the volume of a tetrahedron, whose vertices are any vertex of a given tetrahedron, and the centroids of the faces meeting in that vertex, is $\frac{2}{27}$ of the volume of the given tetrahedron. 56

5897. (C. Leudesdorf, M.A.) — Show that, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$,

$$\left(\frac{a^2-bc}{b+c}+\frac{b^2-ca}{c+a}+\frac{c^2-ab}{a+b}\right)\left(\frac{b+c}{a^2-bc}+\frac{c+a}{b^2-ca}+\frac{a+b}{c^2-ab}\right)=3\frac{a^3+b^3+c^3}{abc}$$

..... 91

5898. (R. A. Graham, M.A.)—A conic is drawn through a point P, and the feet of the normals from it to an ellipse. Show that its centre is the centroid of the points in which a circle of any radius meets the ellipses, the centre of the circle being at P. ... 83

5899. (Gabriel Torelli.)—On donne un triangle par ses trois côtés, et l'on demande l'expression du rapport de sa surface à celle du triangle qui aurait pour sommets les pieds des bisectrices des angles du triangle donné. 85

5900. (J. L. McKenzie, B.A.)—BC, BC′ are two fixed equal chords in a given circle, and A is a fixed point in its circumference; PQ is any chord equal to BC or BC′; the lines BP, AQ intersect in O, and BQ, AP in O′; prove that the loci of O and O′ are circles passing through A and B, one of which touches BC, and the other BC′. 85

5904. (Professor Minchin, M.A.)—Iron filings are scattered over a sheet of paper on which rests a magnet. Prove that the locus of those filings which dip towards the same point in the line of the magnet is a circle. [Magnetism is supposed to be concentrated at the poles.] 73

5906. (Professor Matz, M.A.)—In a given ellipse, determine the average length of all the radius-vectors that can be drawn from one of the foci through the arc intercepted by the focal ordinates. 55

5907. (Professor Wolstenholme, M.A.)—If S_r denote the infinite series $1^{r-1}+2^{r-1}+\frac{3^{r-1}}{2!}+\ldots+\frac{(n+1)^{r-1}}{n!}+\ldots$ to ∞,

prove that $\frac{S_r}{S_1}=\Delta 0^r+\frac{\Delta^2 0^r}{2!}+\frac{\Delta^3 0^r}{3!}+\ldots+\frac{\Delta^r 0^r}{r!}$, a finite series.

..... 88

5911. (Rev. W. A. Whitworth, M.A.)—Show that the 80 acute angles of the 40 triangles of Question 5867 are obtained by rejecting all complete right angles from the values of the expression

$$180°+i\alpha+i'\beta+i''\gamma+i'''\delta;$$

where i, i', i'', i''' may each have any of the values -1, 0, $+1$; and where

$\alpha = 36°\ 52'\ldots = \tan^{-1}\frac{3}{4}$, $\beta = 22°\ 37'\ldots = \tan^{-1}\frac{5}{12}$, $\gamma = 28°\ 4'\ldots = \tan^{-1}\frac{8}{15}$, $\delta = 43°\ 36'\ldots = \tan^{-1}\frac{20}{21}$. 69

5912. (J. J. Walker, M.A.)—Prove that $V.V\beta\gamma V\gamma\alpha$ represents the moment of force β, acting at the end of the vector α, about the (unit) axis γ. 91

No. Page

5913. (C. Taylor, M.A.)—Prove that (1) a normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord; and (2) the polar of the middle point of the chord meets the focal vector to its point of concourse with the directrix upon the normal at its further extremity. 64

5915. (C. Leudesdorf, M.A.)—The four sides of a rectangle ABCD touch a conic, one of whose foci is S; E, F are points on AB, CD respectively, such that the angles CSE, BSF are right angles; show that AD, EF are equidistant from S..... 82

5921. (R. A. Roberts, M.A.)—In the equilibrium of a uniform cord stretched over a smooth sphere and repelled according to the law of the inverse cube of the distance by a uniform circular ring lying on the sphere, if the tension at any point be that to infinity under the action of the force, prove that the catenary will be a circle orthogonal to the ring. 87

5922. (R. E. Riley, B.A.)—Tangents are drawn to an ellipse from points in a concentric circle, radius m; show that the middle points of the chords of contact lie on the curve $r\ (a^2 \sin^2\theta + b^2\cos^2\theta) = a^2b^2m^{-1}$. 92

5923. (D. Edwardes.)—If $x+y+z=0$, prove that, if x, y, z be any real quantities, the expression $x^3y^3+y^3z^3+z^3x^3$ cannot be positive. 78

5924. (Simonelli Ruggero.) — On divise une circonférence en douze parties égales, et l'on joint les points de division au centre. Par l'un de ces points, on abaisse une perpendiculaire sur le rayon voisin; du pied de cette perpendiculaire on abaisse une perpendiculaire sur le rayon suivant, et ainsi de suite. On demande de prouver que la limite de la somme de ces perpendiculaires est égale au diamètre, augmenté du côté du triangle équilatéral inscrit. 84

5937. (Septimus Tebay, B.A.)—Show that, to the first order of approximation, the equation to the section of a comet (consisting of a small spherical nucleus, surrounded by light nebulous matter) by the plane of its orbit is $r = a - b\cos\theta + c\cos 2\theta$; the equilibrium being statical,—an assumption suggested by the general apparition of all comets. 99

5943. (Rev. W. A. Whitworth, M.A.) — If C_r^n denote the number of combinations of n things r together, prove that, whatever be the value of ν,

$$\Sigma_{\alpha=0}^{\alpha=a}\Sigma_{\beta=0}^{\beta=b}\Sigma_{\gamma=0}^{\gamma=c}\cdots\Sigma_{\mu=0}^{\mu=m}\frac{C_\alpha^a C_\beta^b C_\gamma^c \cdots C_\mu^m C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}} = \frac{1+a+b+c+\ldots+m+\ldots}{1+n}$$

..... 95

5944. (H. L. Orchard, L.C.P., &c.)—A spherical soap-bubble is electrified in such a manner that the internal pressure p remains constant. Show that, if ρ_n, ρ_{n+1}, respectively, are the electrical densities when its volume has become n and $n+1$ times its initial volume, we must have the relation

$$\rho_{n+1} : \rho_n = n : (n^2-1)^{\frac{1}{2}} \ldots\ldots 97$$

No. Page

5954. (Rev. Dr. Haughton, F.R.S.)—The general and exact differential equations of motion of an ocean disturbed by any forces, and covering a solid nucleus, of any form, revolving on an axis, being taken as usually written, after LAPLACE; show that the exact equivalents of these equations may be written down at sight, and without calculation, from elementary geometrical and mechanical principles. 103

5956. (Professor Townsend, F.R.S.)—A slender uniform circular ring being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere in its space; show that the attraction:—

(*a*) For all positions of the particle, passes in direction through the vertex of the cone which envelopes along the circumference of the ring its sphere of connection with the position of the particle.

(*b*) For all positions of the particle, varies in magnitude directly as the distance from the centre, and inversely as the square of the product of the greatest and least distances from the circumference of the ring.

(*c*) For positions of the particle limited to a definite sphere passing through the circumference of the ring, varies in magnitude directly as the distance from the fixed centre through which it then passes, and inversely as the square of the distance from the plane of the ring..... 81

CORRIGENDA IN VOL. XXXI.

Page 24, line 7, *for* $27\,(x^2+y^2)^2$ *read* $27c^2\,(x^2+y^2)^2$.

Page 24, line 8, *for* $(x^2+y^2-4c^2)^2$ *read* $(x^2+y^2-4c^2)^3$.

Page 24, line 2 from bottom, *for* $\sin\theta$ *read* $\tan\theta$.

Page 39, line 2, *for* 8 *read* δ, and *for* 28 *read* 2δ.

Page 50, line 6 from bottom, *for* a *read* a_1.

Page 51, line 8 from bottom, *for* they *read* then.

Page 51, line 15 from bottom, *for* $(s-1)^p$ *read* $(s-1\,!)^p$;

Page 56, last line, *for* -2730 *read* $= 2730$.

Page 111, line 14, *for* $+q+xr$ *read* $+qx+r$.

MATHEMATICS

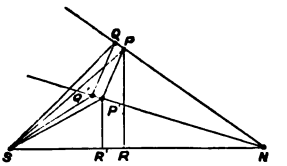
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

5690. (By Professor TOWNSEND, F.R.S.)—Two planets, moving in different planes and describing any orbits round the Sun as common centre of attraction, being supposed to appear stationary in the heavens as observed from each other: if p and p' be their perpendicular distances from the line of nodes of their orbits, and h and h' the synchronous areas of their orbital motions, shew, on elementary principles, that $p : p' = h : h'$.

Solution by the PROPOSER.



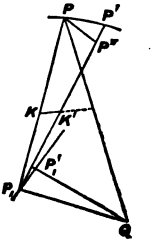
If P and P′ be the positions of the two planets at the instant in question, Q and Q′ their positions at the consecutive instant, R and R′ the feet of the perpendiculars from them upon the line of nodes SN of their orbits, and S the position of the sun; then, since by hypothesis PP′ and QQ′ are parallel to each other, therefore PQ and P′Q′ intersect at some point N in the line of nodes SN of the two orbits, and $PN : P'N = PQ : PQ'$; hence area PSN : area P′SN = area PSQ : area P′SQ′; but area PSN : area P′SN = PR : P′R′ = $p : p'$, and area PSQ : area P′SQ′ = $h : h'$; and therefore, &c.

5766. (By Professor MINCHIN, M.A.)—The vector to any point on a plane curve being ρ, and the successive s-fluxes of ρ being denoted by ρ', ρ'', ρ''', &c., give a simple construction for the direction of the vector ρ'''.

Solution by J. J. WALKER, M.A.

It is known that ρ'' is represented by a vector parallel to the normal—say, at P—and equal to the reciprocal of the radius (R) of curvature. Hence ρ'' will be represented by a vector parallel to the element of the curve which is the locus of the end of a portion of ρ'' equal to that reciprocal; which element will be perpendicular to the line joining the point P with the centre of curvature—say Q—of the evolute at the point corresponding to P. As regards magnitude, it appears without much difficulty, from the geometry of the figure, that $\rho''' = \dot{\rho}'' : \dot{s} = \frac{PQ}{R^2}$.

For, let P′ be consecutive to P, and let P_1, P_1' be the corresponding points on evolute; let $P_1K = \frac{1}{PP'}$, and taking $P'P'' = P_1P_1'$, let $P_1K' = \frac{1}{P'P_1'}$. Then KK′ is parallel to $\dot{\rho}''$; and, in magnitude, $\dot{\rho}'' = \frac{KK'}{\delta t} = \frac{PP''}{R^2\delta t}$.



Now $PP''^2 = PP'^2 + P'P''^2$,

or $P_1P_1'^2 = (PP_1^2 + QP_1^2)(\delta\phi)^2 = (PQ \cdot \delta\phi)^2$,

therefore $\dot{\rho}'' = \frac{PQ}{R^2}\dot{\phi}$;

and finally $\rho''' = \dot{\rho}'' + \dot{s} = \frac{PQ}{R^2}\frac{\dot{\phi}}{\dot{s}} = \frac{PQ}{R^2}$.

As regards direction,

$\tan K' = \tan P'' = PP' : P'P'' = PP' : P_1P_1' = PP_1 : QP_1 = \tan Q$.

Hence KK′ is perpendicular to PQ. [See Hamilton's *Elements*, p. 533, and Tait's *Quaternions*, 1st ed., p. 198.]

5527. (By W. S. B. WOOLHOUSE, F.R.A.S., &c.)—A given triangle may be orthogonally projected from an equilateral triangle; or it may be orthogonally projected into an equilateral triangle. Determine, by an easy geometrical construction, the magnitudes of these equilateral triangles.

I. *Solution by the* PROPOSER.

Let an equilateral triangle PQR, having the equal sides E, be orthogonally projected vertically into the triangle ABC, having the sides a, b, c in a horizontal plane. Then, considering the neutralization of the result of the changes of altitude in passing from Q to R, from R to P, and thence from P to the starting point Q, there arises the relation

$$(E^2 - a^2)^{\frac{1}{2}} + (E^2 - b^2)^{\frac{1}{2}} + (E^2 - c^2)^{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

This equation can subsist only when the three terms of the expression are either all real or all imaginary. In the latter case, the imaginary char-

acter is removed by multiplying by $(-1)^{\frac{1}{2}}$, that is, by writing the equation thus: $(a^2-E^2)^{\frac{1}{2}}+(b^2-E^2)^{\frac{1}{2}}+(c^2-E^2)^{\frac{1}{2}}=0$ ..... (2).

But the form (1) implies that ABC is projected from PQR; and (2) implies that ABC is projected into PQR. Hence, when cleared of radicals, the algebraic relation amongst the sides a, b, c, and E will be identically the same whether the triangle ABC be projected into or projected from the triangle PQR.

If Δ denote the area of the triangle ABC, then

$$16\Delta^2 = 2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4 \quad \ldots\ldots (3).$$

To free (1) or (2) from surds, conceive the three terms of the expression to be the sides of a supposed triangle; then the area of such triangle is obviously zero. Hence, substituting in the formula for $16\Delta^2$, we get

$$0 = 2(E^2-a^2)(E^2-b^2)+2(E^2-a^2)(E^2-c^2)+2(E^2-b^2)(E^2-c^2) \\ -(E^2-a^2)^2-(E^2-b^2)^2-(E^2-c^2)^2;$$

which, arranged according to powers of E, is, by (3),

$$0 = 3E^4-2(a^2+b^2+c^2)E^2+16\Delta^2.$$

Let E_1^2, E_2^2 denote the two roots of this quadratic; then

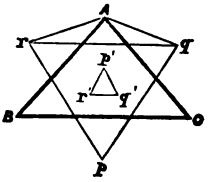
$$E_1^2+E_2^2 = \frac{2(a^2+b^2+c^2)}{3} \quad \text{and} \quad E_1^2E_2^2 = \frac{16\Delta^2}{3};$$

therefore $$(E_1 \pm E_2)^2 = \frac{2(a^2+b^2+c^2)}{3} \pm \frac{8\Delta}{3^{\frac{1}{2}}} \quad \ldots\ldots (4).$$

To effect a geometrical construction of the problem, this result admits of being reduced as follows:—

$$(E_1 \pm E_2)^2 = \frac{4(b^2+c^2)}{3} - \frac{2(b^2+c^2-a^2)}{3} \pm \frac{4bc}{3^{\frac{1}{2}}} \sin A$$
$$= \tfrac{4}{3}\left\{b^2+c^2-bc(\cos A \mp 3^{\frac{1}{2}} \sin A)\right\}$$
$$= \tfrac{4}{3}\left\{b^2+c^2-2bc\cos(A \pm 60°)\right\} \quad \ldots\ldots (5).$$

Now, if p, q, r be the centres of equilateral triangles described externally upon the respective sides BC, CA, AB; and p', q', r' the centres of another set of equilateral triangles described internally, we shall have $Aq^2 = \frac{b^2}{3}$, $Ar^2 = \frac{c^2}{3}$, and the angle $qAr = A+60°$; also $Aq'^2 = \frac{b^2}{3}$, $Ar'^2 = \frac{c^2}{3}$, and the angle $q'Ar' = A-60°$.



Hence $$\left.\begin{matrix} qr^2 \\ q'r'^2 \end{matrix}\right\} = \frac{b^2}{3}+\frac{c^2}{3}-\frac{2bc}{3}\cos(A \pm 60°);$$

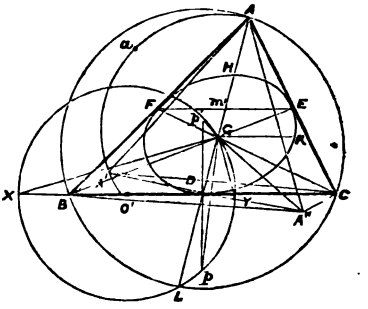
and therefore, by (5), $(E_1 \pm E_2)^2 = \left\{\begin{matrix} 4qr^2 \\ 4q'r'^2 \end{matrix}\right.$, which gives

$$E_1 \pm E_2 = \left\{\begin{matrix} 2qr \\ 2q'r' \end{matrix}\right. \quad \text{and} \quad \left.\begin{matrix} E_1 \\ E_2 \end{matrix}\right\} = qr \pm q'r' \quad \ldots\ldots (6).$$

The symmetry of the expression (4), with respect to a, b, c, shows that the subsidiary triangles pqr, $p'q'r'$ are each of them equilateral; and, by (6), the magnitudes of the sides, E_1, E_2, of the required equilateral triangles are respectively the sum and difference of the sides of these subsidiary equilateral triangles.

II. *Solution by the* PROPOSER.

Let ABC be the given triangle; D, E, F the bisections of the sides BC, CA, AB, respectively; and DEF an inscribed ellipse touching the sides at those points. Then, if the triangle be placed in such a position that the ellipse is projected into a circle, the points of contact will still be at the bisections of the sides, and the projected triangle must necessarily be equilateral and have the diameter of the inscribed circle equal to the minor principal diameter of the ellipse.



The problem is thus resolved into an investigation of a geometrical construction of the magnitudes of the principal diameters of the ellipse inscribed in the given triangle.

As the chord FE bisects AD, it also bisects the semidiameter GH in m; also, since the chord FE is parallel to the tangent BC, it is conjugate to the diameter HD. Let GK, drawn parallel to BC, be the conjugate semidiameter; then $Hm \cdot mD : mE^2 :: GH^2 : GK^2$.

But $Hm = \frac{1}{3}GH$, $mD = \frac{2}{3}GH$; therefore $Hm \cdot mD = \frac{2}{9}GH^2$, and thence $mE^2 = \frac{2}{3}GK^2$, that is, $GK^2 = \frac{3}{2}mE^2 = \frac{1}{3}DC^2$.

Let p, p' be the centres of equilateral triangles described below and upon BC; then $Dp^2 = Dp'^2 = \frac{1}{3}DC^2$. Therefore $GK = Dp = Dp'$.

Now, by Conics, if X, Y be two points in the tangent BC such that $XD \cdot DY = GK^2$, lines drawn from the centre G through these points will determine conjugate diameters of the ellipse; and they will coincide with the principal axes when the angle at G is a right angle, that is, when the circle (O'), described on XY as diameter, passes through G. Since $XD \cdot DY = GK^2 = Dp^2 = Dp'^2$, this circle passes also through the points p, p'.

Again, if AD be produced to meet the circumscribing circle in L, $Dp^2 = \frac{1}{3}DC^2 = \frac{1}{3}AD \cdot DL = GD \cdot DL$; therefore the circle (O') passes also through the point L. Any three of the four known points G, p, p', L will therefore serve to construct the circle (O') geometrically; or the axes GX, GY may be drawn to bisect the interior and exterior angles formed by the known lines Gp, Gp'.

To determine the magnitudes of the principal semi-diameters, α, β, we shall have

$$\begin{aligned} \alpha^2 + \beta^2 &= GD^2 + GK^2 = GD^2 + Dp^2 = \tfrac{1}{2}(Gp^2 + Gp'^2) \\ 2\alpha\beta &= 2GD \cdot GK \sin DGK \quad = 2GD \cdot Dp \cos GDp' \\ &= GD^2 + Dp'^2 - Gp'^2 \quad = \tfrac{1}{2}(Gp^2 + Gp'^2) - Gp'^2 \\ &\qquad\qquad\quad = \tfrac{1}{2}(Gp^2 - Gp'^2). \end{aligned}$$

The sum and difference of these give $(\alpha+\beta)^2 = Gp^2$, $(\alpha-\beta)^2 = Gp'^2$. Hence $2\alpha = Gp + Gp'$, $2\beta = Gp - Gp'$.

The positions and magnitudes of the principal diameters are thus found by an easy construction.

The major diameter (2α) is evidently that of the circle inscribed in the equilateral triangle, from which the given triangle may be projected; and the minor diameter (2β) is that of the circle inscribed in the equilateral

triangle, into which the given triangle can be projected. The magnitudes of the required equilateral triangles are thus completely determined.

The geometrical construction, however, may be effected by a yet more simple and direct method. With vertex A and centre G, suppose an equilateral triangle AA'A'' to be drawn, and let straight lines connect the point A' with B and C. Then the triangles $p'Gp$ and BA'C have the lines GD, A'D respectively bisecting the opposite sides in angles GDp', A'DB, which are equal; moreover, the bisectors have to each other evidently the same ratio as that of the diameter of the inscribed circle to the side of an equilateral triangle, and the bisected sides also have to each other the same ratio. The triangles $p'Gp$, BA'C are therefore similar in all respects, and the two sides Gp, Gp' have to the corresponding sides A'C, A'B the fixed ratio just stated. Hence, as it has been shown that the diameters of the circles inscribed in the required equilateral triangles are equal to the sum and difference of the lines Gp, Gp', it follows that the values of the sides of those triangles are respectively the sum and difference of the lines A'C, A'B; the sum of the said lines being the side of the equilateral triangle from which the given triangle may be orthogonally projected, and the difference of the same lines the side of the equilateral triangle into which the given triangle may be orthogonally projected.

The position of the point A' is easily determinable by the compasses thus: With centre G and radius GA, describe the circle $AaA'\ldots$, and on the circumference lay off the same radius from A to a, and from a to A'. On the other side of AG the point A'' is similarly found. It is obviously immaterial whether the point A' or A'' be used. Also the same results should necessarily follow when either of the other angular points B, C is first used instead of A. This curious theorem is of course involved in the method of construction, and all the points may be simultaneously deduced by conceiving the given triangle to revolve about its centre G through angles equal to one-third and two-thirds of a circumference, and so to occupy the positions A'B'C', A''B''C''.

The axes GX, GY are respectively parallel to the lines which bisect the exterior and interior angles (A') of the triangle BA'C.

NOTE. 1.—The axes GX, GY are the mechanical principal axes of the given triangle ABC considered as a thin lamina. An investigation of their position has always seemed to me to be wanting in works on Dynamics.

NOTE 2. — *Theorem.* — The triangles AB''C', A'BC'', A''B'C, AB'C'', A'B''C, A''BC' are all equilateral.

NOTE 3.—*Theorem.*—By proceeding respectively from B and C as vertices, instead of A, two other circles analogous to (O') are obtained. The centres of the three circles (O') are in a right line; and they all intersect the circumference of the circumscribing circle of the given triangle in one common point, from the perpendiculars upon the sides of the triangle, produced, if necessary, are respectively

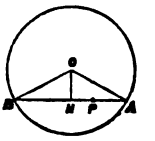
$$\frac{2\Delta a\,(a^2-b^2)\,(a^2-c^2)}{a^2\,(a^2-b^2)\,(a^2-c^2)+b^2\,(b^2-a^2)\,(b^2-c^2)+c^2\,(c^2-a^2)\,(c^2-b^2)},$$

$$\frac{2\Delta b\,(b^2-a^2)\,(b^2-c^2)}{a^2\,(a^2-b^2)\,(a^2-c^2)+b^2\,(b^2-a^2)\,(b^2-c^2)+c^2\,(c^2-a^2)\,(c^2-b^2)},$$

$$\frac{2\Delta c\,(c^2-a^2)\,(c^2-b^2)}{a^2\,(a^2-b^2)\,(a^2-c^2)+b^2\,(b^2-a^2)\,(b^2-c^2)+c^2\,(c^2-a^2)\,(c^2-b^2)}.$$

5721. (By Elizabeth Blackwood.)—A line is drawn at random through a random point within a circle, and another line is drawn at random through the centre; find the chance that these lines will intersect within the circle.

Solution by E. B. Seitz.



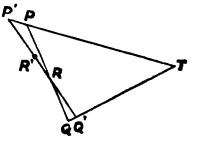
Let O be the centre of the circle, P the random point, and AB a chord through P. Draw OH perpendicular to AB. Now it is evidently necessary to consider only those chords through P which have a given direction. If the random line through O falls in the angle AOB, the two lines will intersect within the circle. Let $OA = r$, $AP = x$, $\angle AOH = \theta$. Then $AB = 2r \sin\theta$; the limits of θ are 0 and $\frac{1}{2}\pi$; and those of x are 0 and $2r \sin\theta$. Hence the required chance is

$$p = \frac{\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta} 4\theta r \sin\theta \, d\theta \, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta} 2\pi r \sin\theta \, d\theta \, dx} = \frac{4}{\pi^2 r}\int_0^{\frac{1}{2}\pi}\int_0^{2r\sin\theta} \theta \sin\theta \, d\theta \, dx$$

$$= \frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi} \theta \sin^2\theta \, d\theta = \frac{1}{2} + \frac{2}{\pi^2}.$$

3560. (By the Rev. A. F. Torry, M.A.)—A chord PQ cuts off a constant area from a given oval curve; show that the radius of curvature of its envelope will be $\frac{1}{4}PQ\,(\cot\theta + \cot\phi)$, θ and ϕ being the angles at which PQ cuts the given curve.

Solution by Professor Nash, M.A.; Prof. Cochez; *and others.*



Let PRQ, P′RQ′ be two positions of the cutting line; then, since the area PRP′ = the area QRQ′, therefore PR = RQ in the limit, and R is a point on the envelope. Similarly R′, the middle point of P′Q′, is on the envelope, and RR′ is an element of the curve. Therefore the radius of curvature is the limit of $\frac{RR'}{\psi}$, where ψ is the angle PRP′.

Now $$\frac{P'R}{PR} = \frac{\sin\theta}{\sin(\theta-\psi)} = 1 + \psi\cot\theta, \text{ in limit;}$$

also $$\frac{Q'R}{QR} = \frac{\sin\phi}{\sin(\phi+\psi)} = 1 - \psi\cot\phi \text{ in limit,}$$

therefore $$2RR' = P'R - Q'R = PR\,\psi\,(\cot\theta + \cot\phi),$$

therefore radius of curvature $= \frac{1}{4}PQ\,(\cot\theta + \cot\phi)$.

5818. (By F. C. Wade, M.A.)—If P_1, P_2, P_3, be the three forces which, acting along the surface of the plane at inclinations of β_1, β_2, β_3 to the line of greatest slope, will support the same weight, prove that

$$P_1 \cos\beta_1 (P_2^2 - P_3^2) + P_2 \cos\beta_2 (P_3^2 - P_1^2) + P_3 \cos\beta_3 (P_1^2 - P_2^2) = 0.$$

Solution by the Rev. G. H. Hopkins, M.A.; R. E. Riley, B.A.; *and others.*

If W, α, μ be respectively the weight of the body, the inclination of the line of greatest slope to the horizon, and the coefficient of friction (F), then $F = \mu W \cos\alpha$. It is noticeable that F is the same for any supporting force, its inclination to the line of greatest slope varying with the supporting force, hence we have the usual relations

$$\mu^2 W^2 \cos^2\alpha = P_1^2 + W^2 \sin^2\alpha + 2P_1 W \sin\alpha \cos\beta_1 \quad \ldots\ldots\ldots\ldots(1),$$
$$\mu^2 W^2 \cos^2\alpha = P_2^2 + W^2 \sin^2\alpha + 2P_2 W \sin\alpha \cos\beta_2 \quad \ldots\ldots\ldots\ldots(2),$$
$$\mu^2 W^2 \cos^2\alpha = P_3^2 + W^2 \sin^2\alpha + 2P_3 W \sin\alpha \cos\beta_3 \quad \ldots\ldots\ldots\ldots(3).$$

Multiplying (1), (2), (3) respectively by $P_2^2 - P_3^2$, $P_3^2 - P_1^2$, and $P_1^2 - P_2^2$, and adding, we obtain the relation

$$P_1 \cos\beta_1 (P_2^2 - P_3^2) + P_2 \cos\beta_2 (P_3^2 - P_1^2) + P_3 \cos\beta_3 (P_1^2 - P_2^2) = 0.$$

5746. (By C. Leudesdorf, M.A.)—The circle of curvature at any point P of the curve $ay^2 = x^3$ cuts the curve in three other points: prove that the locus of the centroid of these three points, as p moves along the curve, is $3ay^2 + (3x + 2a)(7x + 2a)^2 = 0$.

Solution by R. A. Herman, B.A.; E. W. Symons; *and others.*

Let $y = ap^3$, $x = ap^2$; then, if (α, β) be the coordinates of the centre of the osculating circle at a point whose parameter is p, we have

$$\alpha = -\tfrac{1}{2}ap^2(2 + 9p^2), \quad \beta = \tfrac{1}{2}ap(1 + 3p^2), \quad r^2 = \tfrac{1}{16}a^2p^2(4 + 9p^2)^3.$$

The equation to the osculating circle is

$$\{X + \tfrac{1}{2}ap^2(2 + 9p^2)\}^2 + \{Y - \tfrac{1}{2}ap(1 + 3p^2)\}^2 = \tfrac{1}{16}a^2p^2(4 + 9p^2)^3 \quad \ldots(1).$$

Let q be the parameter of the point where it meets the curve, then

$$q^6 + q^4 - \tfrac{1}{2}q^2p(1 + 3p^2) + q^2(2p^2 + 9p^4) = \tfrac{1}{2}p^4 + 2p^6.$$

This equation has 3 equal roots p; hence, dividing out by them, we get

$$q^3 + 3pq^2 + (1 + 6p^2)q + \tfrac{1}{2}(p + 6p^3) = 0.$$

Now if (x, y) be the coordinates of the centroid of the three points where (1) meets the curve, we have

$$x = \tfrac{1}{3}a(q_1^2 + q_2^2 + q_3^2), \quad y = \tfrac{1}{3}a(q_1^3 + q_2^3 + q_3^3),$$
$$q_1^2 + q_2^2 + q_3^2 = 9p^2 - 2(1 + 6p^2) = -(2 + 3p^2),$$
$$q_1^3 + q_2^3 + q_3^3 = -(p + 6p^3) - 3p\{9p^2 - 3(6p^2 + 1)\} = 21p^3 + 8p;$$

therefore $x = -\tfrac{1}{3}a(2 + 3p^2)$, $y = \tfrac{1}{3}a(8p + 21p^3)$,

therefore $3x + 2a = -3ap^2$, $7x + 2a = -\tfrac{1}{3}a(8 + 21p^2)$;

therefore $(7x + 2a)^2(3x + 2a) = -\tfrac{1}{3}a^3p^2(8 + 21p^2)^2 = -3ay^2$;

therefore the locus is $3ay^2 + (3x + 2a)(7x + 2a)^2 = 0$.

5536. (By W. H. H. HUDSON, M.A.)—Trace the curve

$$(x^2+y^2+8c^2)^2(x^2+y^2-c^2) = 108c^2x^2y^2,$$

and prove that its evolute is similar to itself.

Solution by J. HAMMOND, M.A.; Prof. MOREL; *and others.*

Turning the axes round an angle $\frac{1}{4}\pi$, we have

$$(x^2+y^2+8c^2)^2(x^2+y^2-c^2) = 27c^2(x^2-y^2)^2,$$

which becomes, on subtracting from each side $27(x^2+y^2)^2$,

$$(x^2+y^2-4c^2)^2 = -108c^2x^2y^2,$$

the rationalized equation to the quadrantal hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}} = (2c)^{\frac{2}{3}}$.

Assume $x = 2c\cos^3\theta$, $y = 2c\sin^3\theta$; so that $dx : dy = -\cos\theta : \sin\theta$, and the equation to the normal is

$$(y-2c\sin^3\theta)\sin\theta = (x-2c\cos^3\theta)\cos\theta,$$

or

$$x\cos\theta - y\sin\theta = 2c(\cos^2\theta - \sin^2\theta).$$

Differentiating with respect to θ, $x\sin\theta + y\cos\theta = 8c\sin\theta\cos\theta$.

The coordinates of the centre of curvature found by solving these equations are $x = 2c(\cos^3\theta + 3\cos\theta\sin^2\theta)$, $y = 2c(3\cos^2\theta\sin\theta + \sin^3\theta)$.

Therefore $x+y = 2c(\cos\theta+\sin\theta)^3$, $x-y = 2c(\cos\theta-\sin\theta)^3$.

Now put $\frac{x+y}{\sqrt{2}} = X$, $\frac{x-y}{\sqrt{2}} = Y$, and $\theta = \frac{1}{4}\pi - \phi$, and there results

$$X = 4c\cos^3\phi, \quad Y = 4c\sin^3\phi.$$

The evolute of the quadrantal hypocycloid is therefore another quadrantal hypocycloid whose linear dimensions are twice as great, and whose axes are inclined at an angle $\frac{1}{4}\pi$ to those of the given curve.

[The figure of the quadrantal hypocycloid and its evolute is given in BOOTH'S *Geometrical Methods*, Vol. I., p. 156, where this property of the evolute is proved by Boothian Tangential Coordinates.]

3743. (By T. COTTERILL, M.A.)—If a line, length k, rest upon the curve $(xy-ak)^2 = (x+b)^2(k^2-x^2)$ and the line $x = 0$, it envelopes a circle and a parallel to the four-cusped hypocycloid.

Solution by Professor NASH, M.A.; G. TORELLI; *and others.*

Let $y = x\tan\theta + \alpha$ be the equation of the variable line, then the coordinates of the extremity which lies upon the given curve are

$$x = k\cos\theta, \quad y = k\sin\theta + \alpha.$$

Substituting in the equation of the curve, we get

$$(k\alpha\cos\theta + k^2\sin\theta\cos\theta - ak)^2 = (k\cos\theta + b)^2 k^2\sin^2\theta.$$

Dividing by k^2 and putting $\alpha = y - x\sin\theta$,

$$y\cos\theta - x\sin\theta + k\sin\theta\cos\theta - a = \pm(k\sin\theta\cos\theta + b\sin\theta).$$

The upper sign gives $y\cos\theta-(x+b)\sin\theta-a=0$;
the envelope of which is the circle $(x+b)^2+y^2=a^2$. The lower sign gives

$$y\cos\theta-(x-b)\sin\theta+k\sin 2\theta-a=0 \quad \text{.....(A).}$$

The envelope of the line $y\cos\theta-(x-b)\sin\theta+k\sin 2\theta=0$
is the four-cusped hypocycloid $(x-b)^{\frac{2}{3}}+y^{\frac{2}{3}}=(2k)^{\frac{2}{3}}$,
therefore the envelope of (A) is a parallel to this curve.

5141. (By J. J. Walker, M.A.)—Show that the axis of inflexion of the cubic $ax^3+by^3+3fy^2z+3hx^2y+3jzx^2+3kxy^2=0$
is $(af+jk)x+(bj+fh)y+4fjz=0$.

Solution by Prof. Evans, M.A.; Prof. Nash, M.A.; *and others.*

The equation of the Hessian of the given cubic is

$$f^2kx^3+f^2hy^3+f^2jy^2z+(bj^2-2fhj)x^2y+fj^2zx^2+(af^2-2fjk)xy^2=0\ldots(1),$$

therefore the equation

$$(j^3k+\mu a)x^3+(f^2h+\mu b)y^3+(f^2j+3\mu f)y^2z+(fj^2+3\mu j)zx^2$$
$$+(bj^2-2fhj+3\mu h)x^2y+(af^2-2fjk+3\mu k)xy^2=0 \quad \text{.....(2)}$$

represents any cubic passing through all the points of intersection of the Hessian and given cubic.

If this breaks up into linear factors, one of which is the axis of inflexion, the remaining factors must be the common tangents of the Hessian and cubic at the point $x=0$, $y=0$, which is a double point on the given curve. These tangents are represented by $fy^2+jx^2=0$, and if this is a factor of (2) we must have $\mu=fj$, and the remaining factor gives

$$(af+jk)x+(bj+fh)y+4fjz=0.$$

4323. (By Professor Wolstenholme.)—If two points P, Q describe the same circle in the same sense with velocities always in the ratio $m:n$ (m, n being whole numbers prime to each other and $n>m$), the envelope of PQ is an epicycloid whose vertices lie on the given circle and cusps on a concentric circle, whose radius is $\frac{n-m}{n+m}$ times that of the given circle. The class is $n+m$, and the degree $2n$, and the points of contact of tangents drawn from any point on the given circle will be corners of two regular polygons of m sides and n sides respectively. The circular points at infinity are multiple points of the nth degree, of form similar to the origin in the curve $Z^{m+n}\propto X^n$, where Z is the line at infinity and X the straight line joining it to the centre of the given circle.

If the points move in opposite senses, the envelope is an hypocycloid, of degree $2n$ and class $m+n$, the other properties being the same, but for $\frac{n-m}{n+m}$ we write $\frac{n+m}{n-m}$, and for the form of the circular points at infinity, $Z^{n-m} \propto X^m$ or $\propto Y^m$.

Solution by Professor NASH, M.A.

Let O be the centre of the circle, A a point in it at which P and Q meet, $AOP = m\theta$, $AOQ = n\theta$. Then we have to find the envelope of the line

$$x \cos \tfrac{1}{2}(n+m)\theta + y \sin \tfrac{1}{2}(n+m)\theta = a \cos \tfrac{1}{2}(n-m)\theta.$$

Differentiating with respect to θ,

$$x \sin \tfrac{1}{2}(n+m)\theta - y \cos \tfrac{1}{2}(n+m)\theta = \frac{n-m}{n+m} a \sin \tfrac{1}{2}(n-m)\theta.$$

From these equations, we get

$$x = \frac{a}{n+m}(n \cos m\theta + m \cos n\theta), \quad y = \frac{a}{n+m}(n \sin m\theta + m \sin n\theta) \ldots (1),$$

the equations of an epicycloid generated by a circle of radius $\frac{am}{n+m}$ rolling on a circle of radius $a\frac{n-m}{n+m}$, and whose vertices, therefore, lie upon the given circle.

Writing c for $\frac{a}{2(n+m)}$, and t for $\cos\theta + i \sin\theta$, the equations (1) become

$$x = cn(t^m + t^{-m}) + cm(t^n + t^{-n}), \quad iy = cn(t^m - t^{-m}) + cm(t^n - t^{-n}),$$

therefore, to find the points where any line $\lambda x + \mu y + \nu = 0$ meets the curve, we shall get an equation of the degree $2n$ in t; *i.e.*, the degree of the curve is $2n$.

The tangent at any point of the curve is

$$x \cos \tfrac{1}{2}(n+m)\theta + y \sin \tfrac{1}{2}(n+m)\theta + c(n-m)\cos \tfrac{1}{2}(n-m)\theta = 0,$$

or

$$x(t^{\frac{1}{2}(n+m)} + t^{-\frac{1}{2}(n+m)}) + \frac{y}{i}(t^{\frac{1}{2}(n+m)} - t^{-\frac{1}{2}(n+m)}) + c(n-m)(t^{\frac{1}{2}(n-m)} + t^{-\frac{1}{2}(n-m)}) = 0,$$

and if the tangent pass through a given point xy, this equation is of the $(n+m)^{\text{th}}$ degree in t, or the class of the curve is $n+m$.

To find the relations of the curve to the circular points.

The straight line $x + iy = 0$, which joins the centre of the circle to one of the circular points is given by the equation

$$2cnt^m + 2cmt^n = 0.$$

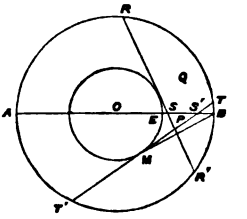
Since this is only of the n^{th} degree, n values of t, and therefore also of x and y, must be infinite; there are also m zero values of t, which give infinite values to x and y.

There are, therefore, $n+m$ points at infinity on this line of which only n lie upon the line at infinity, for otherwise this line would meet the curve in more than $2n$ points. The form therefore is that given by the equation $Z^{n+m} \propto X^n$.

When the points move in opposite directions, the envelope and its properties may be found in the same way as above.

153. NOTE ON THE SOLUTION OF QUESTION 1849. *By* E. B. SEITZ.

The solution of this question in Vol. VIII. of the *Reprint*, by Mr. S. WATSON, is incorrect, as may be seen from the following consideration: Draw the tangent BM, and let P be taken within the space BEM. Through P draw the chords RR′ and TT′ tangent to the concentric circle. Now, according to Mr. WATSON's solution, the chord through P, Q will be less than RR′ as long as Q lies within the space BSR. But if Q lies within the space BS′T, the chord through P, Q will evidently cut the concentric circle, and will consequently be greater than RR′.



By a solution entirely different from Mr. WATSON's, I find $p = \frac{1}{\pi^2}(2\beta - \sin 2\beta)^2$.

5275. (By Professor WOLSTENHOLME, M.A.)—Prove that (1) the locus of the points of intersection of circles of curvature of a parabola drawn at the ends of a focal chord is a bicircular quartic having an acnode and two double tangents; (2) the equation referred to the acnode as origin is $(x^2+y^2-2ax)^2 = 4a^2(3y^2+7x^2)$, that of the parabola being $y^2 = 4a(x+3a)$; (3) the bitangents are inclined at 30° to the axis, and the equation referred to their point of intersection is

$$(r^2 - 14ar\cos\theta + 36a^2)^2 = 4a^2r^2(4\cos^2\theta - 3),$$

the curve being its own inverse with respect to this point. [It would be difficult to show the double tangents in a figure, since for all four values of r to be real we must have $\cos\theta > \frac{1}{2}\sqrt{3}$ and $< \frac{1}{14}\frac{3}{3}$, and the limiting values differ by about 5′ 50″.] (4) The curve osculates the parabola in the two points whose distance from the focus is equal to the latus rectum. (5) The equation of the curve referred to the bitangents is

$$[x^2+y^2+xy-7b(x+y)+12b^2]^2 = 4b^2xy.$$

[Writing b for $a\sqrt{3}$, so that, when $y=0$, $(x-3b)^2(x-4b)^2 = 0$, and if we take $x = \frac{7}{2}b$, we have $(y^2 - \frac{7}{2}by - \frac{1}{4}b^2)^2 = 14b^3y$, and one value of y is nearly $\frac{1}{18}b \div (14 - \frac{7}{4}) = \frac{1}{104}b$. Thus the area included between the curve and a bitangent must be almost invisible on an ordinary scale.] (6) Two circles of curvature touch each other and also this locus at a point whose distances from the node and the centre of inversion are as 5 : 8, being $\frac{10a}{\sqrt{11}}$ and $\frac{16a}{\sqrt{11}}$.

Solution by Professor NASH, M.A.

1. The coordinates of any point on the parabola $y^2 = 4ax$ are $\frac{a}{m^2}, \frac{2y}{m}$; therefore the equation of the circle of curvature is

$$\left(x-2a-\frac{3a}{m^2}\right)^2+\left(y+\frac{2a}{m^3}\right)^2=4a^2\left(\frac{1+m^2}{m^2}\right),$$

or $$m^4(x^2+y^2-4ax)-6axm^2+4aym-3a^2=0,$$

which we may write $Am^4+Bm^3+Cm+D=0.$

The equation of the circle of curvature at the other extremity of the focal chord through (m) is $Dm^4-Cm^3+Bm^2+A=0.$

Eliminating m between these equations, we get for the locus

$$(A+B+D)(A-D)^2+AC^2=0;$$

i.e., $(x^2+y^2-10ax-3a^2)(x^2+y^2-4ax+3a^2)^2+16a^2y^2(x^2+y^2-4ax)=0.$

This equation contains the factor $x^2+(y-a)^2$, giving as part of the locus a pair of imaginary straight lines through the focus of the parabola. Dividing by $x^2+(y-a)^2$, we obtain

$$(x^2+y^2)^2-2a(x^2+y^2)(8x-9a)+3a^2(16x^2-24ax-9a^2)=0.$$

Putting $y=0$, and solving the biquadratic equation, we find

$$x=3a\,.\,3a\,.\,(5\pm2\sqrt{7})\,a,\text{ therefore }3a\,.\,0\text{ is a double point.}$$

2. Transforming to this point at origin, we obtain the equation

$$(x^2+y^2-2ax)^2=4a^2(3y^2+7x^2).$$

The curve represented by this equation is a bicircular quartic, and, the tangents at the origin being given by the equation $y^2+2x^2=0$, are imaginary; therefore the origin is an acnode or conjugate point.

3. The equation may be written in the form

$$(x^2+y^2-2ax-12a^2)^2+4a^2\left\{3y^2-(x+6a)^2\right\}=0,$$

therefore the lines $3y^2-(x+6a)^2=0$ are bitangents, the points of contact lying on the circle $x^2+y^2-2ax-12a^2=0.$

The bitangents make an angle $\tan^{-1}\frac{1}{\sqrt{3}}$ or 30° with the axis, and the points of contact are 0, $\pm 2a\sqrt{3}$, and $-\frac{3}{2}a$, $\pm\frac{1}{2}(3a\sqrt{3})$.

Transforming to the origin $(-6a, 0)$, the equation becomes

$$(x^2+y^2-14ax+48a^2)^2=4a^2(3y^2+7x^2-84ax+252a^2),$$

or $$(x^2+y^2-14ax+36a^2)^2=4a^2(x^2-3y^2),$$

or in polar coordinates

$$(r^2-14ar\cos\theta+36a^2)^2=4a^2r^2(4\cos^2\theta-3).$$

Putting for r, $\frac{36a^2}{r'}$, we get the same equation; therefore the curve is its own inverse with respect to the circle $r=6a$.

4. To find the points where the curve meets the parabola, put $y^2=4a(x+3a)$ in the equation referred to the acnode.

The equation becomes

$$(x^2+2ax+12a^2)^2=4a^2(7x^2+12ax+36a^2),$$

or $$(x^2+2ax+12a^2)^2-4a^2(x+6a)^2=24a^2x^2,$$

i.e., $$x^2(x^2+4ax+24a^2)=24a^2x^2,\text{ or }x^3(x+4a)=0;$$

therefore the curve osculates the parabola at the two points where $x=0$, and the distance of either of these points from the focus is $4a$.

5. If x', y' be coordinates of any point referred to the bitangents

$$x=\tfrac{1}{2}\sqrt{3}\,(x'+y'),\quad y=\tfrac{1}{2}(y'-x');$$

therefore $x^2+y^2 = x'^2+x'y'+y'^2,\ x^2-3y^2 = 3x'y'$.

Substituting in the equation in paragraph (3),

$$\{x'^2+x'y'+y'^2-7a\sqrt{3}(x'+y)+36a^2\}^2 = 12a^2x'y'.$$

Suppressing accents, and writing b for $a\sqrt{3}$, this becomes

$$\{x^2+xy+y^2-7b(x+y)+12b^2\}^2 = 4b^2xy.$$

6. If two circles of curvature touch, the contact must be internal, and the distance between the centres must be equal to the difference of the radii; therefore

$$\left(2a+\frac{3a}{m^2}-2a-3am^2\right)^2+\left(\frac{2a}{m^3}+2am^3\right)^2 = 4a^2(1+m^2)^3\left(1-\frac{1}{m^3}\right)^2.$$

Solving this equation, we get

$$(m^3+1)^3(3m^2-8m+3) = 0, \text{ therefore } m = \tfrac{1}{3}(4\pm\sqrt{7}).$$

When the circles touch, the point of contact divides the line joining the centres externally in the ratio of the radii, *i.e.*, in the ratio $1 : m^3$.

From this we obtain for the coordinates of the point of contact

$$x = \frac{13}{11}a,\quad y = \pm\frac{10\sqrt{7}}{11}a.$$

If r_1, r_2 be distances from node and centre of inversion

$$r_1^2 = \left(3a-\frac{13a}{11}\right)^2+\frac{700a^2}{11^2} = \frac{1100a^2}{11^2}, \text{ therefore } r_1 = \frac{10a}{\sqrt{11}};$$

$$r_2^2 = \left(3a+\frac{13a}{11}\right)^2+\frac{700a^2}{11^2} = \frac{2816a^2}{11^2} = \frac{256a^2}{11}, \text{ therefore } r_2 = \frac{16a}{\sqrt{11}}.$$

5578. (By J. Hammond, M.A.)—Show (1) that the sum to $2m+1$ terms of the series

$$\frac{1}{n}-\frac{2m+n}{n(n+1)}+\frac{(2m+n)(2m+n-1)}{n(n+1)(n+2)}-\frac{(2m+n)(2m+n-1)(2m+n-2)}{n(n+1)(n+2)(n+3)}+\&c.$$

is $(m+n)^{-1}$; and (2) find the sum to infinity when both m and n are fractional, as also $2m+n$.

Solution by Prof. Nash, M.A.; Prof. Evans, M.A; *and others.*

The $(2+1)^{\text{th}}$ term of the series $= u_{2+1}$

$$= (-)^2\frac{(2m+n)(2m+n-1)\ldots(2m+n-2+1)}{n(n+1)\ldots n+2}$$

$$= \frac{(-)^2}{2(m+n)}\left\{\frac{(2m+n)(2m+n-1)\ldots(2m+n-2+1)}{n(n+1)\ldots(n+2-1)}\right.$$
$$\left.+\frac{(2m+n)(2m+n-1)\ldots(2m+n-2)}{n(n+1)\ldots(n+2)}\right\};$$

therefore $\int_{2m+1} = \frac{1}{n} - \frac{1}{2(m+n)}\left\{\frac{2m+n}{n} - \frac{(2m+n)\ldots n}{n(n+1)\ldots(2m+n)}\right\}$

$$= \frac{1}{n} - \frac{m}{n(m+n)} = \frac{1}{m+n}.$$

If m and n be fractional, and $2m = n$, the $(2n+p+1)^{\text{th}}$ term is

$$(-)^p \frac{2n(2n-1)\ldots(-p)}{n(n+1)\ldots(3n+p)} = (-)^p \frac{2n!\,(-)^p p!\,n-1!}{3n+p!},$$

which becomes indefinitely small when p becomes indefinitely large.

$$\int_\infty = \frac{1}{n} - \frac{1}{2(m+n)} \cdot \frac{2m+n}{n} = \frac{1}{n} - \frac{2n}{3n^2} = \frac{1}{3n}.$$

5062. (By W. H. H. Hudson, M.A.)—Trace the curves

$$2\left\{x^2-(y-a)^2\right\}\left\{4x^2-(y-2a)^2\right\} + 9y^2\left\{x^2+a(y-a)\right\} = 0 \quad \ldots\ldots(1),$$

$$a(\alpha^2-\beta\gamma)(\beta+\gamma) = \beta^2\gamma^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Solution by Professor Nash, M.A.

1. Solving the equation as a quadratic in x^2, we get

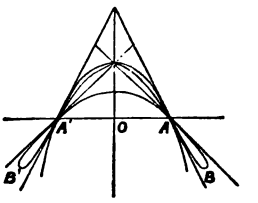
$$16x^2 = y^2 - 24ay + 16a^2 \pm y\left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}}$$

therefore x^2 will be real only when y lies between $\frac{8a}{6\sqrt{2}-3}$, and $-\frac{8a}{6\sqrt{2}+3}$. Putting $x = 0$, we get

$$2(y-a)^2(y-2a)^2 + 9ay^2(y-a) = 0,$$

or $\quad (y-a)(2y-a)(y^2+8a^2) = 0$;

therefore the curve cuts the axis of y at the points $y = \frac{a}{2}$, $y = a$. When $y > a$, all four values of x are imaginary; when $y < a > \frac{a}{2}$,



two are real, and two imaginary; when $y < \frac{a}{2} > -\frac{8a}{6\sqrt{2}+3}$, all are real; when $y < -\frac{8a}{6\sqrt{2}+3}$, all are imaginary.

The curve has no real infinite branches, since the terms of the fourth degree $8x^4 - x^2y^2 + y^4$ have no real factors. Differentiating,

$$32x\frac{dx}{dy} = 2y - 24a \pm \left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}} \pm \frac{24ay - 63y^2}{\left\{(3y+8a)^2 - 72y^2\right\}^{\frac{1}{2}}};$$

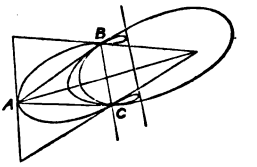
therefore $\frac{dx}{dy} = \infty$, when $x = 0$, or $y = -\frac{8a}{6\sqrt{2}+3}$; and $= -1$ or $-\frac{1}{2}$, when $x = a$, $y = 0$; this point is therefore a double point.

The figure is symmetrical with respect to the axis of y, and the lines

$x = \pm (y-a)$, $2x = \pm (y-2a)$, and the parabola $x^2 + a(y-a) = 0$, determine the compartments within which it must lie.

In the figure A, A′ are nodes, BB′ the bitangent parallel to the axis of x. At the nodes the tangents have contact of the first and second orders respectively.

2. The curve intersects the line at infinity $\alpha+\beta+\gamma = 0$ in imaginary points, and therefore has no infinite branches.



The vertices B and C of the triangle of reference are nodes; the tangents at B are $\gamma = 0$, $\gamma+\alpha = 0$, for when $\gamma = 0$, $\alpha^3\beta = 0$; and three consecutive points lie on the tangent, which, however, is only a single tangent, since B is a node.

$\gamma+\alpha = 0$ meets the curve in four consecutive points, and is therefore an inflexional tangent. The line $(\beta+\gamma = 0)$ meet the curve in four consecutive points at A, and therefore has contact of the third order with it.

Any line $\alpha = k(\beta+\gamma)$ parallel to $\alpha = 0$ will meet the curve in four points given by the equation

$$k^3 \left\{\frac{(\beta+\gamma)^2}{\beta\gamma}\right\}^2 - k\frac{(\beta+\gamma)^2}{\beta\gamma} - 1 = 0, \text{ or } \frac{(\beta+\gamma)^2}{\beta\gamma} = \frac{1 \pm (1+4k)^{\frac{1}{2}}}{2k^2}.$$

If $4k = -1$, we have $\frac{\beta}{\gamma} = 3 \pm 2\sqrt{2}$, and $4\alpha+\beta+\gamma = 0$ is a bitangent.

If $4k < -1$ all the points are imaginary. If $4k > -1$, and less than the value determined by the real root (different from zero) of the equation $\frac{1 \pm (1+4k)^{\frac{1}{2}}}{2k^2} = 4$, all the roots are real, and for greater values two are real and two imaginary. For this value of k two of the points coincide, and are situated on the line $\beta = \gamma$.

The lines $\alpha = 0$, $\beta+\gamma = 0$, and the conic $\alpha^2 - \beta\gamma = 0$, determine the compartments within which the curve lies.

5792. (By the Rev. Dr. HAUGHTON, F.R.S.)—Show that, within the limits of NEWTON'S construction for the disturbing force, the *exact equation* of the surface of an equatorial canal of constant width and depth, attracted by the moon always in the equator, is $\rho = -a\cos 2\phi + b$; also trace the form of the curve for the following particular cases:—(1) $b = 0$; (2) $b < a$; (3) $b = a$; (4) $b > a$, but $< 5a$; (5) $b =$ or $> 5a$.

I. *Hints towards a Solution; by the* PROPOSER.

1. My coordinates (ρ, ϕ) are the common (r, θ) coordinates, and do not refer to the radius of curvature and angle made by it with a fixed line.

2. My solution is *exact* (within the limits of NEWTON'S construction for

the disturbing force), that is, to terms of the first order of $\left(\frac{a}{R}\right)$, where a is the radius of the earth and R the distance of the moon.

3. The solution of my problem requires the conception of a *Platonic form*, like that of a river, where the water, at each several place along the bank, always behaves in the same way peculiar to that place, although the raindrops composing it are always different.

4. The Newtonian construction for the tangential disturbing force is expressed by the equation $f = -k \sin 2\phi$, where $k = \frac{1}{200000}$ ft. per second.

II. *Solution by* Professor Townsend, F.R.S.

Denoting by z the depth and by v the velocity of the water at any point (r, θ) of the surface of the canal (the coordinates r and θ being supposed measured, the former from the centre of the earth, and the latter from its line of connection with that of the moon, in the direction of the earth's rotation); by z_0, v_0, and r_0 the mean values of z, v, and r; by m the mass of the moon, by d its distance from the earth's centre, and by T the tangential component of its disturbing action at $r\theta$, which, within the limits of Newton's construction,

$$= -\frac{3}{2}\frac{m}{d^3} r_0 \sin 2\theta ;$$

then, since approximately, within the limits in question,

$$v dv = T ds = -\frac{3}{2}\frac{m}{d^3} r_0^2 \sin 2\theta \, d\theta,$$

and therefore, by integration,

$$v^2 = v_0^2 + \frac{3}{2}\frac{m}{d^3} r_0^2 \cos 2\theta,$$

and since, to the same degree of approximation, from the incompressibility of the water,

$$vz = \text{const.} = v_0 z_0,$$

and therefore, by substitution in the preceding equation,

$$\frac{v_0^2 z_0^2}{z^2} = v_0^2 + \frac{3}{2}\frac{m}{d^3} r_0^2 \cos 2\theta ;$$

herefore approximately, within the same limits,

$$z = z_0\left[1 - \frac{3}{4}\frac{m}{d^3}\frac{r_0^2}{v_0^2}\cos 2\theta\right]$$

$$= z_0\left[1 - \frac{3}{4}\frac{m}{d^3}\frac{1}{\omega^2}\cos 2\theta\right],$$

where ω is the velocity of the earth's rotation, and consequently

$$r = r_0 - \frac{3}{4}\frac{m}{d^3}\frac{z_0}{\omega^2}\cos 2\theta ;$$

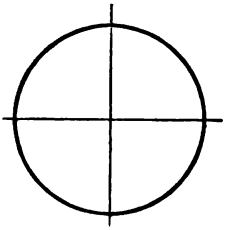


Fig. 1.

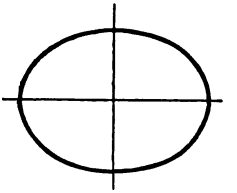


Fig. 2.

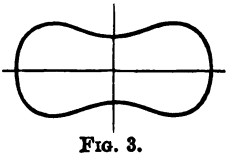


Fig. 3.

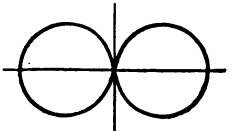


Fig. 4.

which is Dr. HAUGHTON'S equation, and exact within the limits of NEWTON'S construction.

The tracing of the biaxal curve

$$r = a + b \cos 2\theta,$$

for different relative values of a and b, presents several interesting varieties of form, of which the following are the principal:—For $b = 0$ it is manifestly a circle of radius a (Fig. 1), respecting which, of course, nothing further need be noticed. For b small in comparison with a, as is the case in Dr. HAUGHTON'S equation, it is a biaxal oval of semi-axes $a + b$ and $a - b$, somewhat resembling an ellipse in appearance (Fig. 2), which form it retains in general character during the increase of the ratio of b to a, until that ratio becomes $= \frac{1}{2}$. For $b > \frac{1}{2}a$ but $< a$, the oval changes into the dumb-bell form (Fig. 3), and retains it in general character during the increase of the ratio from $\frac{1}{2}$ to 1. For $b = a$ the curve assumes the figure of eight form (Fig. 4), passing cuspidally twice through its centre, and touching cuspidally twice its then evanescent second axis of figure. For $a < b$ but > 0, it assumes the corolla form (Fig. 5), passing four times without inflexion through its centre, and touching at each passage one of two central tangents symmetrically situated with respect to both its axes of figure. And, finally, for $a = 0$, the four loops become equal (Fig. 6), the corolla becomes perfect, and the two common tangents at the centre to its four petals intersect at right-angles and bisect the angles between the axes of the figure. The above obviously exhaust all the real varieties of form of the curve.

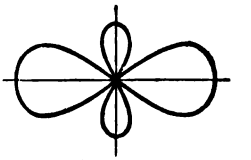


FIG. 5.

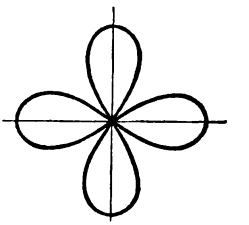


FIG. 6.

III. *Solution by* J. J. WALKER, M.A.

Let P be any particle of the water, T the earth's centre, TA a fixed equatorial radius, S the moon's centre, and M her accelerating force at unit distance; also let $\angle \mathrm{STA} = \alpha$ (a function of the time), $\angle \mathrm{PTA} = \phi$, $\mathrm{PT} = \rho$. The difference of the components of Moon's force on P and T in the direction TP, and perpendicular to PT, respectively, are

$$-k\frac{a}{\mathrm{R}}, \text{ and } k\frac{3a \sin 2(\alpha-\phi)}{2\mathrm{R}}, \text{ where } k = \frac{\mathrm{M}}{\mathrm{R}^2}.$$

Write gravity $= -kg'$. Then, if p is the hydrodynamic pressure at P,

$$\frac{dp}{d\rho} + \frac{d^2\rho}{dt^2} = -k\left(g' + \frac{a}{\mathrm{R}}\right), \quad \frac{dp}{d\phi} + \frac{a^2d^2\phi}{dt^2} = k\frac{3a^2 \sin 2(\alpha-\phi)}{2\mathrm{R}}.$$

Neglecting $\frac{d^2\rho}{dt^2}$ and $\frac{a^2d^2\phi}{dt^2}$ as inconsiderable (or, the latter being disregarded, and the former being treated, as by EULER, as proportional to $a-\rho$, $\frac{dp}{d\rho}d\rho + \frac{dp}{d\phi}d\phi$ would be an exact differential), and multiplying by $d\rho$ and by $d\phi$ respectively, adding and integrating, t being treated as constant,

$$p + \mathrm{C} = k\left\{-\left(g' + \frac{a}{\mathrm{R}}\right)\rho + \frac{3a^2 \cos 2(\alpha-\phi)}{4\mathrm{R}}\right\}.$$

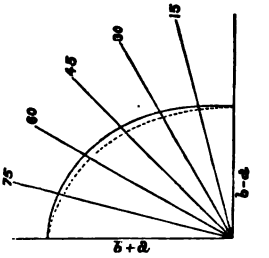
For the instantaneous free surface put $p = C'$, and the equation is of the form given in the question.

As regards the form of the curve $\rho = -a \cos 2\phi + b$.

1. This case has been often discussed. It is of the class of "Flores Geometrici," having four equal loops like petals radiating from the pole.

2. Also a curve of two pairs of loops with axes $\pm (a \mp b)$. In these two cases the curve passes four times through the pole without inflexion.

3. An 8-shaped curve, the pole being a double cuspidal point—the prime radius, the cuspidal tangent.



4. A dumb-bell shaped curve, having axes $= 2(b \pm a)$. It is convex towards the pole at the ends of the less, concave at the ends of the greater axis. The directions of the real points of inflexion are given by

$$2\cos^2\phi = -m + 1 + (m^2+2)^{\frac{1}{2}},$$

where m lies between 1 and 5, between which limits the dexter lies between 1 and 0; *e. g.*, when $m = \frac{1}{4}$, $\phi = \frac{1}{12}\pi$ nearly. For values of m greater than this, ϕ diminishes rapidly.

5. There is no real inflexion, and the figure differs but little from that of an ellipse with the same semi-axes $(b+a, b-a)$, being somewhat flatter at the ends of the axes, and consequently lying at other points outside the ellipse. A trace of a quadrant of the figure for the value $b = 6a$ is annexed, the inner dotted line being the contour of the coaxal ellipse.

5748. (By J. J. Walker, M.A.)—If xyz satisfy $ax + \beta y + \gamma z = 0$, and $u = 0$, u being a ternary form of the order p, show how to transform

$$\beta\gamma^2 y \left\{ \frac{d^2u}{dx^2}\left(\frac{du}{dy}\right)^2 + \frac{d^2u}{dy^2}\left(\frac{du}{dx}\right)^2 - 2\frac{d^2u}{dx\,dy}\frac{du}{dx}\frac{dx}{dy} \right\}$$
$$+ \beta^2\gamma z \left\{ \frac{d^2u}{dx^2}\left(\frac{du}{dz}\right)^2 + \frac{d^2u}{dz^2}\left(\frac{du}{dx}\right)^2 - 2\frac{d^2u}{dz\,dx}\frac{du}{dz}\frac{du}{dx} \right\}$$

into the symmetrical form

$$\alpha\beta\gamma \left\{ (p-1)\frac{du}{dx}\frac{du}{dy}\frac{du}{dz} + x\frac{d^2u}{dy\,dz}\left(\frac{du}{dx}\right)^2 + y\frac{d^2u}{dz\,dx}\left(\frac{du}{dy}\right)^2 + z\frac{d^2u}{dx\,dy}\left(\frac{du}{dz}\right)^2 \right\}.$$

Solution by the Proposer; A. Buchheim, Ph.D.; *and others.*

Writing l, m, n for $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, and a, b, c, f, g, h for $\frac{d^2u}{dx^2} \ldots \frac{d^2u}{dx\,dy}$, since $xl + ym + zn = 0$,

$$\begin{aligned} yam^2 + ybl^2 - 2yhlm &= -xalm - zamn + ybl^2 + xhl^2 + zhnl - yhlm \\ &= -(p-1)l^2m + zglm + zamn + (p-1)l^2m - zfl^2 + zhnl \\ &= -z\{amn + l(fl - gm - hn)\}. \end{aligned}$$

Similarly,

$$\alpha n^2 + \gamma l^2 - 2\alpha\gamma nl = -y\{amn + l(fl - gm - hn)\}.$$

The expression to be transformed is therefore equal to

$$-\beta\gamma(\gamma z + \beta y)\{amn - l(-fl + gm + hn)\},$$

or $$\alpha\beta\gamma\{(ax + hy + gz)\, mn + xfl^2 - xglm - zgmn - xhnl - yhmn\},$$

or $$\alpha\beta\gamma\{(p-1)\, lmn + xfl^2 + ygm^2 + zhn^2\}.$$

3633. (By the Rev. G. H. Hopkins, M.A.)—From any point on the surface of a circular cylinder planes are drawn; find the equation to the surface upon which the foci of all the elliptic sections are placed, and prove that the section of this surface by a plane through the fixed point, and containing the axis of the cylinder, will be the logocyclic curve.

Solution by Professor Nash, M.A.

Let the equation of the cylinder be $x^2 + y^2 = 2ax$, and the equation of any plane through the origin $lx + my + nx = 0$ .....(1). The major axis of the section of the cylinder by the plane is the intersection of (1) and $\frac{x-a}{l} = \frac{y}{m}$, therefore the equations of the axis are

$$\frac{x-a}{l} = \frac{y}{m} = \frac{lx + my - al}{l^2 + m^2} = -\frac{nz + al}{l^2 + m^2},$$

and the lengths of the axes are $\left(\frac{2a}{n},\ 2a\right)$; hence the coordinates of the foci are given by $\frac{x-a}{nl} = \frac{y}{lm} = -\frac{z + aln^{-1}}{l^2 + m^2} = \pm\frac{a}{n}$.

Eliminating l, m, n, we get

$$\frac{z(2ax - x^2 - y^2)^{\frac{1}{2}}}{a} \pm (x-a) = \mp\frac{(x-a)^2 + y^2}{a},$$

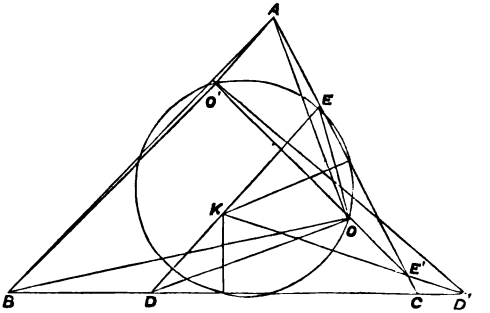
or $$z^2(2ax - x^2 - y^2) = \{y^2 + x(x-a)\}^2.$$

The section of this surface by the plane $y = 0$, which contains the axis of the cylinder, and the given point is $z^2(2a - x) = x(x - a,)^2$ which is the equation of the logocyclic curve.

5834. (By J. J. Walker, M.A.)—Referring to Quest. 4372 (*Reprint*, Vol. XXIII., p. 30), show that, if OD, OE, OF, perpendicular respectively to OA, OB, OC, meet BC, CA, AB respectively in D, E, F (three collinear

points, as Prof. KELLAND has shown); then, if O is a point on the circumference passing through the middle points of BC, CA, AB, the transversal DEF passes through the centre of the circle circumscribing ABC.

Solution by the PROPOSER.



If α, β, γ are vectors from O to the points A, B, C, and ω the vector to the centre of the circumference through ABC, and

if $\gamma = l\alpha + m\beta$,

then (Quest. 5774),

if $a = \mathrm{T}(\beta - \gamma)$,

$b = \mathrm{T}(\gamma - \alpha)$,

$c = \mathrm{T}(\alpha - \beta)$;

$\mathrm{S}_1 = \mathrm{S}(\gamma - \alpha)(\alpha - \beta)$,

$\mathrm{S}_2 = \mathrm{S}(\alpha - \beta)(\beta - \gamma)$,

$\mathrm{S}_3 = \mathrm{S}(\beta - \gamma)(\gamma - \alpha)$,

$$(a^2\mathrm{S}_1 + b^2\mathrm{S}_2 + c^2\mathrm{S}_3)\,\omega = (a^2\mathrm{S}_1 + lc^2\mathrm{S}_3)\,\alpha + (b^2\mathrm{S}_2 + mc^2\mathrm{S}_3)\,\beta \ldots\ldots\ldots\ldots (1).$$

Also, if δ, ϵ are the vectors OD, OE, and $\mathrm{S} = \mathrm{S}\beta\gamma$, $\mathrm{S}' = \mathrm{S}\gamma\alpha$, $\mathrm{S}'' = \mathrm{S}\alpha\beta$,

$$(\mathrm{S}' - \mathrm{S}'')\,\delta = -l\mathrm{S}''\alpha + (\mathrm{S}' - m\mathrm{S}'')\beta,\quad (\mathrm{S}'' - \mathrm{S})\,\epsilon = (-\mathrm{S} + l\mathrm{S}'')\alpha + m\mathrm{S}''\beta \ldots (2, 3).$$

Eliminating α, β from (1), (2), (3), and equating with zero the sum of the coefficients of ω, δ, ϵ in the result,

$$(1 - l - m)\,(a^2\mathrm{S}'\mathrm{S}''\mathrm{S}_1 + b^2\mathrm{S}''\mathrm{S}\mathrm{S}_2 + c^2\mathrm{S}\mathrm{S}'\mathrm{S}_3) = 0.$$

Rejecting $1 - l - m = 0$, the condition of A, B, C being collinear, the second factor, expressed in terms of α, β, $\mathrm{S}\alpha\beta$, l, m, is equal to

$$(\nabla\alpha\beta)^2 \left\{2lm\mathrm{S}\alpha\beta + l(l-1)\alpha^2 + m(m-1)\beta^2\right\}\left\{(1 - l^2 - m^2)\,\mathrm{S}\alpha\beta - lm(\alpha^2 + \beta^2)\right\};$$

of which factors, the vanishing of the first is the condition that OA, OB shall be in the same straight line; of the second that O shall be in the circumference through ABC (and thus the Quaternion Solution of 4372 is completed); while the vanishing of the third is the condition that O shall be in the circumference passing through the middle points of BC, CA, AB (see Question 5803). The annexed figure represents two positions of O (O, O') on the nine-point circumference, and shows the collinearity of the points D, E (or D', E') with K the centre of the circumference through A, B, C.

5793. (By Professor SYLVESTER, F.R.S.)—

If $u_i = \left\{\left(\frac{d}{dx}\right)^i f(x)\right\}^2 - \frac{\mathrm{A} + \mathrm{B}i - \mathrm{B}}{\mathrm{A} + \mathrm{B}i}\left\{\left(\frac{d}{dx}\right)^{i-1} f(x) \cdot \left(\frac{d}{dx}\right)^{i+1} f(x)\right\}$,

prove that, when $u_i = 0$, $u_{i+1} = 0$, ..., $u_{i+w} = 0$,

$$\frac{\left(\frac{d}{dx}\right)^w u_i}{u_{i+w}} = \frac{\mathrm{A} + \mathrm{B}(i+w)}{\mathrm{A} + \mathrm{B}i} \cdot \frac{\left(\frac{d}{dx}\right)^i f(x)}{\left(\frac{d}{dx}\right)^{i+w} f(x)}.$$

Solution by ROBERT RAWSON.

Put D for $\left(\frac{d}{dx}\right)$, and v for $f(x)$; then, assuming the property in the question to be true for w, it can be shown to be true for $w+1$, as follows:

By the question, we have

$$u_{i+w} = (\mathrm{D}^{i+w}v)^2 \quad - \frac{\mathrm{A}+\mathrm{B}(i+w-1)}{\mathrm{A}+\mathrm{B}(i+w)} \cdot \mathrm{D}^{i+w-1} \cdot v\mathrm{D}^{i+w+1} \cdot v \ldots\ldots(1),$$

$$u_{i+w+1} = (\mathrm{D}^{i+w+1} \cdot v)^2 - \frac{\mathrm{A}+\mathrm{B}(i+w)}{\mathrm{A}+\mathrm{B}(i+w+1)} \cdot \mathrm{D}^{i+w} \cdot v\mathrm{D}^{i+w+2} \cdot v \ \ldots\ldots(2).$$

When $u_{i+w} = 0$, then

$$(\mathrm{D}^{i+w} \cdot v)^2 = \frac{\mathrm{A}+\mathrm{B}(i+w-1)}{\mathrm{A}+\mathrm{B}(i+w)} \mathrm{D}^{i+w-1} \cdot v\mathrm{D}^{i+w+1} \cdot v \ \ldots\ldots\ldots(3).$$

Differentiating (1) with respect to x, and using equation (3), we have

$$\mathrm{D}u_{i+w} = 2\mathrm{D}^{i+w}v\mathrm{D}^{i+w+1} \cdot v - \frac{\mathrm{A}+\mathrm{B}(i+w-1)}{\mathrm{A}+\mathrm{B}(i+w)} \times (\mathrm{D}^{i+w}v\mathrm{D}^{i+w+1}v + \mathrm{D}^{i+w-1}v\mathrm{D}^{i+w+2}v)$$

$$= \frac{\mathrm{A}+\mathrm{B}(i+w+1)}{\mathrm{A}+\mathrm{B}(i+w)} \mathrm{D}^{i+w}v\mathrm{D}^{i+w+1}v - \frac{\mathrm{A}+\mathrm{B}(i+w-1)}{\mathrm{A}+\mathrm{B}(i+w)} \mathrm{D}^{i+w-1}v\mathrm{D}^{i+w+2}v$$

$$= \frac{\mathrm{A}+\mathrm{B}(i+w+1)}{\mathrm{A}+\mathrm{B}(i+w)} \cdot \frac{\mathrm{D}^{i+w}v}{\mathrm{D}^{i+w+1}v} \left\{ (\mathrm{D}^{i+w+1}v)^2 - \frac{\mathrm{A}+\mathrm{B}(i+w)}{\mathrm{A}+\mathrm{B}(i+w+1)} \mathrm{D}^{i+w}v\mathrm{D}^{i+w+2}v \right\} \ldots(4).$$

From (2) and (4), $\frac{\mathrm{D}u_{i+w}}{u_{i+w+1}} = \frac{\mathrm{A}+\mathrm{B}(i+w+1)}{\mathrm{A}+\mathrm{B}(i+w)} \cdot \frac{\mathrm{D}^{i+w} \cdot v}{\mathrm{D}^{i+w+1} \cdot v} \ldots\ldots\ldots\ldots(5).$

Assume $\mathrm{D}^w \cdot u_i = \frac{\mathrm{A}+\mathrm{B}(i+w)}{\mathrm{A}+\mathrm{B}i} \cdot \frac{\mathrm{D}^i \cdot v}{\mathrm{D}^{i+w} \cdot v} u_{i+w} \ \ldots\ldots\ldots\ldots(6);$

differentiate (6), and put $u_{i+w} = 0$, then

$$\mathrm{D}^{w+1} \cdot u_i = \frac{\mathrm{A}+\mathrm{B}(i+w)}{\mathrm{A}+\mathrm{B}i} \cdot \frac{\mathrm{D}^i \cdot v}{\mathrm{D}^{i+w} \cdot v} \cdot \mathrm{D}u_{i+w} \ \ldots\ldots\ldots\ldots(7).$$

From (7) and (5), $\frac{\mathrm{D}^{w+1} \cdot u_i}{u^{i+w+1}} = \frac{\mathrm{A}+\mathrm{B}(i+w+1)}{\mathrm{A}+\mathrm{B}i} \cdot \frac{\mathrm{D}^i \cdot v}{\mathrm{D}^{i+w+1} \cdot v} \ldots\ldots\ldots\ldots(8).$

Hence, if the property be true for w, it is true for $w+1$. The property is true when $w = 0$, then it is true for $w = 1$, &c.

5852. (By Professor MINCHIN, M.A.)—Any rigid body is kinetically equivalent to three equal uniform spheres whose centres are the vertices of a maximum triangle inscribed in a certain ellipse lying in one of the principal planes at the centre of gravity of the body. [From WOLSTENHOLME'S *Problems*, 2nd ed., Ex. 2601.]

Prove this geometrically from the fact that, if two intersecting rectangular lines of constant lengths revolve in a fixed plane about their point

of intersection, a point whose abscissa and ordinate are the projections of these lines on any fixed line, will trace out an ellipse.

Solution by the PROPOSER.

Let the principal radii of gyration at the centre of gravity O of the body be k_1, k_2, k_3, and let three uniform spheres, the mass of each being one-third of that of the body, have their centres at the points $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$, the radius of each being r. (The axes of coordinates are the principal axes of the body.)

Then the conditions for kinetic equivalence are

$$x_1+x_2+x_3=0,\quad y_1+y_2+y_3=0,\quad z_1+z_2+z_3=0 \ldots\ldots\ldots\ldots(1);$$

$$x_1y_1+x_2y_2+x_3y_3=0,\quad y_1z_1+y_2z_2+y_3z_3=0,\quad z_1x_1+z_2x_2+z_3x_3=0 \ldots (2);$$

$$\left.\begin{aligned} \tfrac{2}{5}r^2+x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2 &= 3k_3^2,\\ \tfrac{2}{5}r^2+x_1^2+x_2^2+x_3^2+z_1^2+z_2^2+z_3^2 &= 3k_2^2,\\ \tfrac{2}{5}r^2+y_1^2+y_2^2+y_3^2+z_1^2+z_2^2+z_3^2 &= 3k_1^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(3).$$

Assume three points, P, Q, R, whose coordinates are respectively $(x_1 x_2 x_3)$, $(y_1 y_2 y_3)$, $(z_1 z_2 z_3)$. Then equations (1) show that these points all lie in a plane equally inclined to the axes of coordinates, the cosine of the angles of inclination being $\sqrt{\frac{1}{3}}$. Also, equations (2) show that the three lines OP, OQ, OR (which are coplanar, observe) are mutually rectangular. This is impossible unless one of them vanishes. Suppose OR = 0; that is, $z_1 = z_2 = z_3 = 0$. Hence the centres of the spheres all lie in the plane xy.

Again, equations (3) give $\frac{2}{5}r^2 = 3\,(k_1^2+k_2^2-k_3^2)$,

and also $\quad \mathrm{OP}^2 = 3\,(k_3^2-k_1^2),\quad \mathrm{OQ}^2 = 3\,(k_3^2-k_2^2)$.

Now x_1 and y_1 are the projections of OP and OQ along the axis of x; and a point, whose coordinates are the projections of OP and OQ along a line making an angle θ with the plane in which OP and OQ revolve, describes the ellipse $\quad \dfrac{x^2}{\mathrm{OP}^2}+\dfrac{y^2}{\mathrm{OQ}^2}=\cos^2\theta.$

Hence, since $\cos^2\theta = \frac{2}{3}$ in this case, $\quad \dfrac{x_1^2}{k_3^2-k_1^2}+\dfrac{y_1^2}{k_3^2-k_2^2}=2,$

together with the very same equation for $(x_2 y_2)$ and $(x_3 y_3)$; and equations (1) show that the centres are the vertices of a triangle whose centroid is the centre of mass of the body.

5387. (By Professor CAYLEY, F.R.S.)—Show that a cubic surface has at most 4 conical points, and a quartic surface at most 16 conical points.

Solution by J. HAMMOND, M.A.

By reasoning precisely similar to that in SALMON'S *Higher Curves*, we see that the degree of the reciprocal must be diminished by 2 for each

conical point on the surface. Hence the degree of the reciprocal of a surface of degree m, having 8 conical points, is $m(m-1)^2-28$. [See SALMON's *Geometry of Three Dimensions*, Art. 275, 2nd ed.] Thus a cubic with 4 conical points and a quartic with 16 conical points reciprocate into quartics with 4 and 16 singular tangent planes respectively. And they cannot have 5 conical points in the case of the cubic, or 17 in the case of the quartic; for then they would reciprocate into surfaces of the second degree.

---

154. NOTE ON LANDEN'S THEOREM; *by* CHRISTINE LADD.

That part of the proof of LANDEN'S Theorem which is given in paragraph 54 of Professor CAYLEY'S *Elliptic Functions* can be simplified as follows:—

Reverting to the original two circles, if in the figure $\angle APO = \theta$ $(= \frac{1}{2}\pi+\phi-\psi)$, then obviously $AA' \cos MAO = AM d\theta$, that is,

$$R \cdot 2d\phi \sin AOG = AG\, 2d\phi = AM d\theta;$$

hence the equation $\frac{d\phi}{AM} = -\frac{d\psi}{BM}$ may be completed into

$$\frac{d\phi}{AM} = -\frac{d\psi}{BM} = \frac{d\theta}{2AG};$$

and, observing that $AG^2 = AO^2 - OG^2 = R^2 - (D\sin\theta - r)^2$,

the equation is $$\frac{d\phi}{\Delta(k,\phi)} = \frac{d\theta\,[(R+D)^2-r^2]^{\frac{1}{2}}}{2\,[R^2-(D\sin\theta-r)^2]^{\frac{1}{2}}}.$$

We have $OG = R\sin OAG$; but $\angle OAG = \phi+\psi-\frac{1}{2}\pi = 2\phi-\theta$;

hence $$D\sin\theta - r = R\sin(2\phi-\theta),$$

which is an integral equation corresponding to the above differential equation.

5849. (By Sir JAMES COCKLE, F.R.S.)—Find the general solution of the linear partial differential equation of the first order

$$\frac{n}{z}(z^m + Ax^r + B)(p+q) = \frac{n}{n-r}\left(\frac{dA}{dx}+\frac{dA}{dy}\right)x^r + \frac{dB}{dx}+\frac{dB}{dy} \quad \ldots\ldots(1),$$

wherein n, m, and r are unequal, and A and B are free from z, and are not functions one of the other.

I. *Solution by the* PROPOSER.

Generally (taking $n > m$) let the summation S refer to r, and be made from $r = m$ to $r = 0$, and let the summation Σ refer to a and x, so that

$\Sigma ax = a_1x_1 + a_2x_2 + \ldots + a_lx_l$ wherein $x_1, x_2, \ldots x_l$ are independent variables, and the sum $(a_1 + a_2 + \ldots + a_l)$ of the constants vanishes, *i. e.*, $\Sigma a = 0$. Then we shall have

$$\Sigma \frac{d}{dx} \Sigma ax = \left(\frac{d}{dx_1} + \frac{d}{dx_2} + \ldots + \frac{d}{dx_l}\right)\Sigma ax = \Sigma a = 0,$$

and $\Sigma \frac{dz}{dx}$ will represent $\frac{dz}{dx_1} + \frac{dz}{dz_2} + \ldots + \frac{dz}{dx_l}$.

Let $\phi\Sigma ax$ be an arbitrary function of Σax, and let

$$z^n - (\phi\Sigma ax)\left(S \frac{nA_r}{n-r} z^r\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)};$$

then, operating on (2) with $\frac{z}{n}\Sigma\frac{d}{dx}$, we have

$$\left\{z^n - (\phi\Sigma ax)\left(S \frac{rA_r}{n-r} z^r\right)\right\} \Sigma \frac{dz}{dx} - z(\phi\Sigma ax)\left(S \frac{1}{n-r} \Sigma \frac{dA_r}{dx} z^r\right) = 0,$$

for

$$\Sigma \frac{d}{dx} \phi\Sigma ax = 0.$$

From the result of the operation eliminate z^n by means of (2), and divide out $\phi\Sigma ax$; we shall have

$$SA_r z^r \Sigma \frac{dz}{dx} - zS \frac{z^r}{n-r} \Sigma \frac{dA_r}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

and, as (ii.) is a general solution of (i.), so is

$$z^n - \phi(y - x)\left\{\frac{nz^m}{n-m} + \frac{nAz^r}{n-r} + B\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

a general solution of (1); viz., of

$$\frac{n}{z}(z^m + Az^r + B)(p + q) = \frac{n}{n-r}\left(\frac{dA}{dx} + \frac{dA}{dy}\right) z^r + \frac{dB}{dx} + \frac{dB}{dy} \ldots\ldots(1).$$

5819. (By Prince CAMILLE DE POLIGNAC.)—On each side of a triangle as segment, an internal arc of a circle is drawn containing an angle of 120°; (1) prove that the coordinates of the common point of intersection of the three circles referred to the given triangle are proportional to

$$\frac{1}{\sin A + \sqrt{3}\cos A}, \quad \frac{1}{\sin B + \sqrt{3}\cos B}, \quad \frac{1}{\sin C + \sqrt{3}\cos C};$$

and (2) find a characteristic feature of the point in relation to the triangle.

I. *Solution by the* PROPOSER.

1. Write the three circles $MN = x^2$, $N'L = y^2$, $L'M' = z^2$; then, ρ, σ, τ being constant multipliers, we find, attending to data,

$$\sigma M \equiv z \sin 60° - x \sin (B - 60°), \quad \tau N \equiv y \sin 60° - x \sin (C - 60°);$$

or putting, for shortness' sake,

$$\sin 60° = K, \quad \sin(A-60°) = \alpha, \quad \sin(B-60°) = \beta, \quad \sin(C-60°) = \gamma,$$
$$\sigma M = Kz-\beta x, \quad \tau N' = Kx-\gamma y, \quad \rho L' = Ky-\alpha z,$$
$$\tau N = Ky-\gamma x, \quad \rho L = Kz-\alpha y, \quad \sigma M' = Kx-\beta z.$$

Take $\rho = \sin A$, $\sigma = \sin B$, $\tau = \sin C$; and it will be easily seen that M, N, &c., represent the actual lengths of the coordinates, the rule of signs being duly attended to. Writing out, we find, for the first circle,

$$(\beta\gamma-\sigma\tau)x^2+K^2yz-K\gamma zx-K\beta xy = 0.$$

Putting this into the form $(ln+my+nz)(\rho x+\sigma y+\tau z)+h(\rho y^2+\sigma zx+\tau xy)$, to which the equation of any circle may be brought (SALMON'S *Conics*), we find $m = n = 0$, $l = \frac{\beta\gamma-\sigma t}{\rho}$, $h = \frac{K^2}{\rho}$; whence,

for the first circle, $(\beta\gamma-\sigma\tau)(\rho x+\sigma y+\tau z)x+K^2(y\rho z+\sigma zx+\tau xy) = 0$;

and for the other two, through a circular permutation,

$$(\gamma\alpha-\tau\rho)(\rho x+\sigma y+\tau z)y+K^2(\rho yz+\&c.) = 0,$$
$$(\alpha\beta-\rho\sigma)(\rho x+\sigma y+\tau z)z+K^2(\rho yz+\&c.) = 0.$$

The radical axes are

$$(\beta\gamma-\sigma\tau)x = (\gamma\alpha-\tau\rho)y, \quad (\gamma\alpha-\tau\rho)y = (\alpha\beta-\rho\sigma)z, \quad (\alpha\beta-\rho\sigma)z = (\beta\gamma-\sigma\tau)x,$$

and they meet in a point the coordinates of which can be written

$$x = \frac{1}{\beta\gamma-\sigma\tau}, \quad y = \frac{1}{\gamma\alpha-\tau\rho}, \quad z = \frac{1}{\alpha\beta-\rho\sigma}.$$

Writing at full length, we find

$$\frac{1}{x} = \sin(B-60°)\sin(C-60°)-\sin B\sin C$$
$$= \tfrac{1}{2}\{\cos(B-C)-\cos(B+C-120°)-\cos(B-C)+\cos(B+C)\}$$
$$\frac{1}{x} = \tfrac{1}{2}-\{\cos A+\cos(A-60°)\},$$

whence, on expanding and dropping constant factors, we find the result stated in the question.

The point will lie inside the triangle, coincide with a vertex, or lie outside, according as every angle $< 120°$, or one of them $= > 120°$.

2. The fundamental property is the following. Take any point M inside triangle ABC. Place MA, MB, MC $= \alpha, \beta, \gamma$ respectively; then $\alpha+\beta+\gamma$ is a minimum when M coincides with the point just found. This appears at once from the fact that, if an ellipse be described with B, C as foci, and a circle be drawn with A as a centre to touch the ellipse, the point of contact M will give a minimum sum, $\alpha+\beta+\gamma$, with respect to all the points of the ellipse. The same remark applies to an ellipse and circle foci C, A and centre B, or foci A, B and centre C. Consequently the minimum-minimorum occurs at a point such that each vector, α, β, or γ, bisects the angle between the other two. In other words, inside the triangle, $\angle BMC = \angle CMA = \angle AMB = 120°$.

We can extend the notion of minimum to the algebraical sum $\alpha+\beta+\gamma$, reckoning each vector, say $\alpha = MA$, as negative, where the x of M is negative, and so on, *mutatis mutandis*. Working with hyperbolas as well as with ellipses, we shall infer one more point where each vector bisects internally or externally the angle under the other two. It will be given likewise as the intersection of three circles of which the sides of the

triangle are chords subtending 120° externally, and its coordinates are

$$x = \frac{1}{\sin A - \sqrt{3}\cos A}, \quad y = \frac{1}{\sin B - \sqrt{3}\cos B}, \quad z = \frac{1}{\sin C - \sqrt{3}\cos C}.$$

When A = B the two minima-points lie on the bisector of ∠C. When A = B = C = 60°, the three external circles coincide with the circumscribing circle, and the minimum point, the limit of their intersection, depends in position on the mode of degenerescence of the triangle. But, as it must lie on the bisector of one of the angles, it can only coincide with one of the intersections of the bisectors with the circumscribing circle at which $\alpha + \beta + \gamma = 0$.

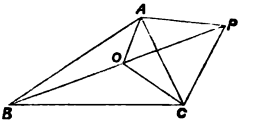
The locus of minima is simply the locus of the feet of normals drawn from a fixed point to a system of confocal conics, or, which is the same, the locus of points of contact of tangents drawn from the same point. An easy description of it is the following. For example, let A be the fixed point, and B, C the foci. Then draw through A any radius vector AR. From B let fall a perpendicular BP on it, and produce it a length PB′ = BP. Then the line B′C meets AR in a point M, which belongs to the locus. For, obviously, M is the vertex of an isosceles triangle MBB′, the two sides of which pass through B and C respectively, while AM is the internal or external bisector of ∠M. This mode of description will readily disclose the chief characteristics of the curve, which is a circular cubic, as is, I suppose, well known. It will be found to pass through A, B, C with a node in A, the tangents being the two bisectors of ∠A. The asymptote is parallel to the bisector of the side BC drawn through A, &c. The Cartesian equation, where A is the origin, follows immediately, being, in fact, $u_2 + u_3 = 0$, where u_2 = product of bisectors of ∠A, and $u_3 = lU$, l being the bisector of the side BC, while U is determined by the condition that the curve should pass through B and C. The trilinear equation will be found to be $S_1 \equiv 2\cos B \cdot y^2z - 2\cos C \cdot z^2y + x(y^2 - z^2) = 0$.

By permuting circularly A, B, C and x, y, z, S_2 and S_3 will be obtained, and it will appear that $S_1x + S_2y + S_3z \equiv 0$,

showing that the three cubics pass through the same points of intersection, agreeably to former conclusions. These points are to be reckoned as follows:—

The two circular points	...	...	2	Total 9.
The three vertices	...	...	5	
The two minima points	...	...	2	

II. *Solution by* JOHN YOUNG, B.A.



Let O be the point whose coordinates are required. Join AO, BO, CO, and produce BO to meet the circle COA again in P. The angles AOP, COP each contain 60°, and are respectively equal to ACP, CAP, by reason of the circle APCO; from which it follows that the triangle ACP is equilateral, hence PC = PA = b. Also angle BCP = C + 60°, BAP = A + 60°; and perpendiculars from P on BA, BC produced are respectively

$$b \sin (A + 60°), \quad b \sin (C + 60°).$$

Hence equation of line BO is

$$\alpha \sin (A + 60°) = \gamma \sin (C + 60°).$$

Similarly, equation of line CO is

$$\beta \sin (B + 60°) = \alpha \sin (A + 60°).$$

Thus, O is determined by the equations

$$\alpha \sin (A + 60°) = \beta \sin (B + 60°) = \gamma \sin (C + 60°);$$

or, expanding and substituting values for sin 60°, cos 60°,

$$\alpha : \beta : \gamma = \frac{1}{\sin A + \sqrt{3} \cdot \cos A} : \frac{1}{\sin B + \sqrt{3} \cdot \cos B} : \frac{1}{\sin C + \sqrt{3} \cdot \cos C}.$$

The sum of the lines OA, OB, OC is a minimum (THOMSON'S *Euclid*, Appendix).

5861. (By HUGH MCCOLL, B.A.)—Required a solution of the following logical problem:—Suppose it to have been ascertained by observation (1) that, whenever the events A and B happen together, they are invariably followed by the event C, and also by either the event D or the event E; and (2) that, whenever the events D and E both happen, they have invariably been preceded by the event A, or else by both the events B and C. When may we conclude (from the occurrence or non-occurrence of the events A, B, C, D)—(1) that E will certainly happen; (2) that E will certainly *not* happen?

I. *Solution by* Prof. LLOYD TANNER, M.A.

Any combination of the occurrences and non-occurrences of A, B, C, D, E, may be represented by a symbol such as ABC'D'E, where the accents indicate non-occurrences. Write out the complete set of combinations, 32 in number; from these erase those excluded by the conditions of the problem. Of the remainder, some are in pairs, such as A'BCD'E, A'BCD'E'; whence the combination A'BCD' tells nothing as to the occurrence of E. Besides these pairs, we have ABCD'E without the corresponding E' symbol, and A'B'C'DE', A'BC'DE', A'B'CDE' without the corresponding E symbol. Hence E will happen if A, B (and C) happen, but D does not happen; E will not happen if D happens, and either A and B or A and C do not happen. This may serve as a verification of the results otherwise obtained.

II. *Solution by the* PROPOSER.

Let the statement a assert the occurrence of the event A, with any additions or limitations as to time, place, or manner, necessary to distinguish the event A from any other event that might be confounded with it; and let a' be the denial of a, asserting the non-occurrence of A under the stated limitations. Let b, b', c, c', &c., be similarly interpreted. The

data of the problem will then be the two implications,

$$ab : c(d+e), \quad de : a+bc.$$

Solving these implications with respect to e and e', which, in this problem, may be done by mere inspection, we get

$$ab(c'+d') : e, \quad abc'+da'(b'+c') : e'.$$

The term abc', being common to both antecedents, is zero, from which we learn that ab implies c, symbolically expressed by the implication $ab : c$. Omitting this zero term, we get

$$abd' : e, \quad da'(b'+c') : e'.$$

This agrees with the result otherwise obtained by Professor TANNER.

[To anyone who has read Mr. MCCOLL's second paper on the *Calculus of Equivalent Statements*, published in the *Proceedings* of the London Mathematical Society, the above solution will be perfectly clear as it stands, without a stroke of extra working.]

5832. (By the EDITOR.)—In BOOTH's *Geometrical Methods*, Vol. I., p. 114, it is shown that the sections of the surface of centres (S) by a principal plane of the ellipsoid are an ellipse and the evolute of an ellipse. Prove that the principal plane *intersects* on one sheet of the surface in the evolute section, and *touches* the other sheet in the elliptic section. And, generally, that the normals at two consecutive points of a line of curvature of the ellipsoid *intersect* on one sheet of the surface of centres, while their plane touches the other sheet.

Solution by H. STABENOW, M.A.

Since the evolute section of S by a principal plane of the ellipsoid *abc*, for instance by the xy-plane, is the evolute of the ellipse *ab*, it follows that the normals N and N′ to the latter at any two of its consecutive points, A and B, intersect on the former, and consequently on one sheet of S, "as two consecutive tangents to it;" and that their plane, that is the principal plane, necessarily intersects that same sheet.

Further, there exists generally, at each point of a given surface, a couple of principal sections, and also of lines of curvature, cutting each other at right angles; therefore—

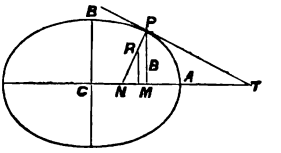
1. The elliptic section of S is the locus of the extremities of the radii of curvature at A, B, &c., of the ellipses V, V′, &c., resulting from the intersection of the ellipsoid by planes through the normals N, N′, &c., and perpendicular to the ellipse or the xy-plane.

2. Through each point, as A, of the ellipse *ab*, and its consecutive in the vertical section V, passes a line of curvature.

From this, and that the locus of the principal centres, answering to the different points of a line of curvature, is on the surface S a curve, to which the corresponding radii of curvature are the consecutive tangents, it is plain that any two consecutive normals, as N and N′, while each intersects the elliptic section of the surface, are at two consecutive points of this section tangents to the surface itself; and that consequently the plane of these normals, that is the principal plane, must touch the surface.

In like manner, it may be shown that the normals at two consecutive points of a line of curvature enjoy the property enunciated in the theorem.

The equation of the elliptic section of the surface S can be obtained directly as follows:—



Considering (see fig.) any point P (α, β) of the ellipse ab, and designating the conjugate of the diameter through P by b'; the distance of the centre C to the tangent PT by $p = \frac{ab}{b'}$; the sections of the ellipsoid (ellipses) by planes passing through the normal PN and the ordinate PM $(= \beta)$, and perpendicular to the plane of the ellipse ab, respectively, by V and E; the radii of curvature at P of V and E respectively by PR and ρ; the angle between PN and PM, or between V and E, by γ; and the coordinates of the point R by x, y:

then since $\rho = \frac{c^2(a^2-\alpha^2)^{\frac{1}{2}}}{ab}$ and $\cos\gamma = \frac{a^2(a^2-\alpha^2)^{\frac{1}{2}}}{a^4-(a^2-b^2)\alpha^2} = \frac{(a^2-\alpha^2)^{\frac{1}{2}}}{a(a^2-e^2\alpha^2)}$,

where $e^2 = \frac{a^2-b^2}{a^2}$, we find, by a known theorem, for the value of PR,

$$\mathrm{PR} = \frac{\rho}{\cos\gamma} = \frac{c^2(a^2-e^2\alpha^2)}{ab} = \frac{c^2b'}{ab} = \frac{c^2}{p},$$

and consequently $\frac{\mathrm{PN}}{\mathrm{PR}} = \frac{\left(\frac{b^2}{p}\right)}{\left(\frac{c^2}{p}\right)} = \frac{b^2}{c} = \frac{\beta}{\beta-y} = \frac{\frac{b^2}{a^2}\alpha}{\alpha-x}$;

from which we deduce $\frac{\alpha}{a} = \frac{ax}{a^2-c^2}$, $\frac{\beta}{b} = \frac{by}{b^2-c^2}$.

Substituting these values of $\frac{\alpha}{a}$, $\frac{\beta}{b}$ in the equation of the ellipse $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$, we get, for that of the elliptic section of the surface S,

$$\frac{a^2x^2}{(a^2-c^2)^2} + \frac{b^2y^2}{(b^2-c^2)^2} = 1.$$

---

155. Note on Mr. Stabenow's Solution of Question 5832.

By J. Hammond, M.A.

The first part of the solution is perfectly general, and holds for any surface whatever, and its surface of centres. It is, in fact, identical with Art. 302 of Salmon's *Geometry of Three Dimensions* (2nd Ed., p. 238). But Salmon's proof of the theorem in question is so clear, that it is worth giving in his own words.

"Now if from a point, not on a surface, be drawn two consecutive tangent lines to the surface, the plane of those lines is manifestly a tangent plane to the surface; for it is a tangent plane to the cone which is drawn from the point touching the surface. But if two consecutive tangent lines intersect on the surface, it cannot be inferred that their plane touches the

surface. For if we cut the surface by any plane whatever, any two consecutive tangents to the curve of section (which, of course, are also tangent lines to the surface) intersect on the curve, and yet the plane of these lines is supposed not to touch the surface.

"Consider now the two consecutive normals at the points M, N, these are both tangents to both sheets of the surface of centres. And since the point C, in which they intersect, is on the first sheet, but not necessarily on the second, the plane of the two normals is the tangent plane to the second sheet of the surface of centres."

In BOOTH'S *Geometrical Methods* the evolute section in the plane of xy is obtained by putting $\zeta = 0$ in the tangential equation of the surface of centres. This shows that the tangent planes perpendicular to the plane of xy envelope a cylinder whose base is the evolute of the ellipse (a, b).

The elliptic section, on the other hand, is obtained by putting $\zeta = \infty$; *i. e.*, making the plane of xy a tangent plane.

Thus the plane of xy intersects the surface of centres at right angles in the evolute section, and touches it along the elliptic section.

5223. (By Professor TOWNSEND, F.R.S.) — A uniform cylindrical elastic rod, supported at both extremities and at both points of trisection by four rigid props in the same horizontal right line, being supposed in strained equilibrium, with slight deflection of form, under the action of gravity; required the distribution of its weight between the props, with the points of inflexion of its strained axis arising from their reactions.

I. *Solution by* Professor BALL, F.R.S.

Let E be the coefficient of elasticity.

Let I be the moment of inertia of a vertical section of the rod about a horizontal line drawn in the plane of section through the neutral axis.

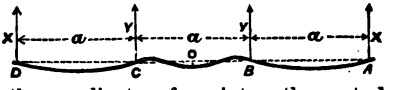
Let X be the reaction of either of the extreme props.

Let Y be the reaction of either of the intermediate props.

Let Δ be the deflection of the central point of the neutral.

Let $3a$ be the length of the axis of the entire rod.

Let w be its load per running foot.



Extreme segment.—Let x, y be the coordinates of a point on the neutral axis of the right-hand segment AB of the rod, the origin of coordinates being at the extremity A; then, from the well-known formulæ, we have

$$\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{X}x - \tfrac{1}{2}wx^2,$$

whence, by integration, $\mathrm{EI}y = \frac{1}{6}\mathrm{X}x^3 - \frac{1}{24}wx^4 + \mathrm{C}x + \mathrm{C}'$;

but $y = 0$ if $x = 0$ or $x = a$, whence finally,

$$\mathrm{EI}y = \tfrac{1}{6}\mathrm{X}(x^3 - a^2x) + \tfrac{1}{24}(a^3x - x^4)w \quad \ldots\ldots\ldots\ldots (1),$$

$$\mathrm{EI}\frac{dy}{dx} = \mathrm{X}(\tfrac{1}{2}x^2 - \tfrac{1}{6}a^2) + (\tfrac{1}{24}a^3 - \tfrac{1}{6}x^3)w \quad \ldots\ldots\ldots\ldots (2).$$

If $x = a$, we have $\left(\frac{dy}{dx}\right)_{x=a} = \frac{I}{EI}(\frac{1}{3}a^2X - \frac{1}{6}a^3w)$ ..... (3).

Central segment.—As the whole weight of the rod is equal to the sum of the reactions at the points of support, we have $Y + X - \frac{3}{2}aw$.

Taking the origin of coordinates at the right-hand point of trisection B, we have, for the equation of the central segment,

$$EI\frac{d^2y}{dx} = Yx + X(a+x) - a(\tfrac{1}{2}a+x)w - w(\tfrac{1}{2}x^2).$$

Integrating and reducing, we find

$$EIy = -\tfrac{1}{2}aXx + \tfrac{5}{24}a^3wx + \tfrac{1}{4}aXx^2 - \tfrac{1}{12}a^2wx^2 + \tfrac{1}{12}awx^3 - \tfrac{1}{24}wx^4 \text{.....(4)},$$

and, when $x = 0$, we have $\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{EI}(\frac{5}{24}a^3w - \frac{1}{2}a^2X)$ .....(5).

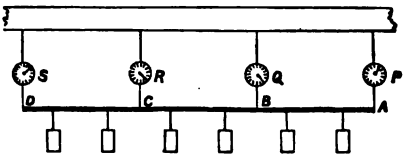
As the curvature of the neutral axis is not abruptly altered when the rod crosses the supports, we must have the values of $dy \div dx$ in the equations (3) *and* (4) *equal;* whence

$$Pa^2X - 3a^3w = 5a^3w - 12a^2X, \quad \text{or} \quad X = \tfrac{3}{5}aw.$$

The entire load L being $3aw$, we have finally

$$X = \tfrac{4}{10}L; \quad Y = \tfrac{11}{10}L.$$

Experimental verification.—This result was verified by a simple experiment. A pine rod AD, four feet long and half-an-inch square, is suspended by four dynamometers P, Q, R, S in the manner shown in the figure, the distances AB, BC, CD being equal. Six weights of 14 lbs. each were hung from the beam, at equal distances apart, so as to produce approximately the effect of a continuous load, and the dynamometers were adjusted by the wire strainers from which they depended, so that the four points of support should be in a horizontal line.



The total load of the rod, including its own weight and also the weights of the iron *s*-hooks by which the load was attached, was 86 lbs. According to the theory, therefore, the strain on the extreme dynamometers should be each 11·5 lbs., and that borne by the two interior should be 31·5 lbs. The dynamometers were graduated to read to single pounds, and the two extreme ones showed 10 lbs. and 11 lbs. respectively, while the two interior ones marked 32, 33. The accordance is quite as close as could be expected. The slight discrepancy is due to the circumstance that the condition of uniform loading is only approximately fulfilled.

Transformation of the equations.—If in equation (4) we make $x = a + 2$, and substitute for X its value $2aw + 5$, we obtain, for the deflection at the centre, the expression $\Delta = \frac{wa^4}{1920EI}$.

Introducing this value into the equation of the central segment, *and transforming the origin of coordinates to the centre* O *of the line* AD, we

obtain finally the equation of the neutral axis

$$y = \Delta\left(2\frac{x}{a}-1\right)\left(2\frac{x}{a}+1\right)\left((20)^{\frac{1}{2}}\frac{x}{a}-1\right)\left((20)^{\frac{1}{2}}\frac{x}{a}+1\right) \ldots (6).$$

For the extreme segment of the rod with the origin still at the extremity A, we have, for the neutral axis, the equation

$$y = 16\Delta\left(3\frac{x}{a}-8\frac{x^3}{a^3}+5\frac{x^4}{a^4}\right) \ldots\ldots\ldots\ldots (7).$$

Deflection in the extreme segment.—If we make $dy \div dx = 0$ in equation (7), we obtain the cubic $3a^3 - 24ax^2 + 20x^3 = 0$, of which the roots are $+1{\cdot}0689a$, $+0{\cdot}44617a$, $-0{\cdot}31478a$. It is plain that the root we require is $+0{\cdot}44617a$, and this is, therefore, the abscissa of the point of greatest deflection in the extreme segment. Introducing this value for x into the equation (7), we find, for the deflection of the lowest point, the value $13\cdot2\Delta$. It thus appears that the deflection in the extreme segments is more than thirteen times as great as the deflection at the centre.

The point of inflexion in the extreme segment is found by making $d^2y \div dx^2 = 0$ in (7) to be $x = \frac{4}{5}a$.

Deflection in the central segment.—The maximum and minimum deflection are easily seen to occur at the points $x = (\frac{3}{20})^{\frac{1}{2}}\,a$, $x = 0$, $x = -(\frac{3}{20})^{\frac{1}{2}}\,a$, the corresponding deflections being $y = -\frac{4}{5}\Delta$, $y = +\Delta$, $y = -\frac{4}{5}\Delta$. By making $d^2y \div dx^2 = 0$, we find that the points of inflexion in the central segment are identical with the points in which the neutral axis cuts the horizontal axis.

Summary.—Suppose a girder 300 feet long supported in the manner we have been considering. Let Δ be the deflection at the centre O. The neutral axis intersects the horizontal axis at a point 22 feet from O, and it is also remarkable that this is a point of inflexion. At a distance of 38 feet the neutral axis culminates, attaining an elevation of $4\Delta \div 5$ *above the horizontal axis*. From this point the neutral axis approaches the horizontal axis, and crosses it at a point of support 50 feet from O. At this point the neutral axis is inclined to the horizontal axis by an angle equal to $16\Delta \div 100$. The neutral axis now descends rapidly below the horizontal axis. A second point of inflexion is reached at a distance of 70 feet from O, where the deflection is $5{\cdot}6\Delta$; from this the deflection increases to a maximum amount of $13{\cdot}2\Delta$, at a distance of 106 feet from O. This is the lowest point of the neutral axis, which finally ascends to meet the horizontal axis at an angle of $48\Delta \div 100$ in the terminal point of support.

II. *Solution by the* Proposer.

Taking as axes, of x and of y respectively, the horizontal line of the props and the vertical bisector of the rod, and denoting, respectively, by a and b the semi-intervals between the extreme and intermediate pairs of props, by P and Q the equal values of the corresponding pairs of reactions, by W the entire weight of the rod, by E its ordinary modulus of longitudinal elasticity, and by I its horizontal moment of sectional inertia; we have, on the ordinarily received principles, for any point xy on either extreme, and for any point $x'y'$ on the mean segment of its strained axis,

the approximate equations of equilibrium,

$$\text{EI} \cdot \frac{d^2y}{dx^2} = \text{P}(a-x) - \frac{1}{4}\frac{\text{W}}{a}(a-x)^2 \quad \ldots\ldots(1),$$

$$\text{EI} \cdot \frac{d^2y'}{dx'^2} = \text{P}(a-x') + \text{Q}(b-x') - \frac{1}{4}\frac{\text{W}}{a}(a-x')^2 \quad \ldots\ldots(2),$$

which, if P and Q were known, would give at once, by integration, the form and all other particulars of the equilibrium of the rod under the circumstances of the case.

By the first integration, remembering that $\frac{dy'}{dx'} = 0$ when $x' = 0$, and $= \frac{dy}{dx}$ when $x' = x = b$, we get at once, from (1) and (2),

$$\text{EI} \cdot \frac{dy}{dx} = \tfrac{1}{2}\text{P}[a^2-(a-x)^2] + \tfrac{1}{2}\text{Q}b^2 - \frac{1}{12}\frac{\text{W}}{a}[a^3-(a-x)^3] \quad \ldots\ldots(3),$$

$$\text{EI} \cdot \frac{dy'}{dx'} = \tfrac{1}{2}\text{P}[a^2-(a-x')^2] + \tfrac{1}{2}\text{Q}[b^2-(b-x')^2] - \frac{1}{12}\frac{\text{W}}{a}[a^3-(a-x')^3] \ldots(4),$$

which give the values of the angular deflection at all points of both segments when P and Q are known.

By the second integration, remembering that $y = 0$ when $x = a$, and $= y'$ when $x = x' = b$, we get again, from (3) and (4),

$$\text{EI} \cdot y = -\tfrac{1}{2}(\text{P}a^2 + \text{Q}b^2 - \tfrac{1}{3}\text{W}a^2)(a-x) + \tfrac{1}{3}\text{P}(a-x)^3 - \frac{1}{48}\frac{\text{W}}{a}(a-x)^4 \ldots\ldots(5),$$

$$\text{EI} \cdot y' = -\tfrac{1}{2}(\text{P}a^2 + \text{Q}b^2 - \tfrac{1}{3}\text{W}a^2)(a-x') + \tfrac{1}{3}\text{P}(a-x')^3 + \tfrac{1}{3}\text{Q}(b-x')^3 - \frac{1}{48}\frac{\text{W}}{a}(a-x')^4 \ldots\ldots(6),$$

which give the values of the linear deflection at all points of both segments when P and Q are known.

To determine the values of P and Q, and thus complete the solution of the question. Since y or $y' = 0$ when x or $x' = b$, we get at once, from (5) or (6), the linear relation between them

$$(\text{P}a^2 + \text{Q}b^2 - \tfrac{1}{3}\text{W}a^2) - \tfrac{1}{3}\text{P}(a-b)^2 + \frac{1}{24}\frac{\text{W}}{a}(a-b)^3 = 0 \ldots\ldots(7),$$

which, combined with the evident relation $\text{P} + \text{Q} = \tfrac{1}{2}\text{W}$, gives immediately for P and Q the general values

$$\text{P}(a-b)[(a+b) - \tfrac{1}{2}(a-b)] = \frac{1}{6}\frac{\text{W}}{a}[a^3 - 3ab^2 - \tfrac{1}{4}(a-b)^3] \ldots\ldots(8),$$

$$\text{Q}(a-b)[(a+b) - \tfrac{1}{2}(a-b)] = \frac{1}{6}\frac{\text{W}}{a}[a^3 + 2a^2b - ab^2 + \tfrac{1}{4}(a-b)^3] \ldots(9);$$

whatever be the ratio of the semi-intervals a and b between the extreme and intermediate pairs of props, provided, of course, that the intervals themselves, as all along supposed, are concentric.

When $a = 3b$, as supposed in the question, the general relations (8) and (9) give for P and Q the particular values

$$\text{P} = \tfrac{4}{10}\text{W}, \text{ and } \text{Q} = \tfrac{1}{10}\text{W} \ldots\ldots(10),$$

which, accordingly, is the required distribution of the weight of the rod between the two pairs of props.

To find the points of inflection, if any, of the extreme segments of the

rod. Since for them $\frac{d^2y}{dx^2} = 0$, therefore, at once, from (1) and (10),

$$(a-x)^2 - \tfrac{2}{15}a(a-x) = 0 \quad \ldots\ldots (11),$$

which, rejecting the solution $x = a$, corresponding to the obvious inflections at the terminal props, gives $x = \frac{1}{15}a$ for the distance of the other point of inflection of either extreme segment from the centre of the rod.

To find the points of inflection, if any, of the mean segment of the rod. Since for them $\frac{d^2y'}{dx'^2} = 0$, therefore, at once, from (2) and (10),

$$(a-x')^2 - 2a(a-x') + \tfrac{44}{45}a^2 = 0 \quad \ldots\ldots (12),$$

which gives $x' = \pm\frac{1}{3}\frac{a}{\sqrt{5}}$ for the equal distances, in opposite directions, of the two points of inflection of the mean segment from the centre of the rod.

To find the points of greatest linear deflection of the mean segment of the rod. Since for them $\frac{dy'}{dx'} = 0$, therefore, from (4) and (10), after a few trifling reductions,

$$x'^3 - \tfrac{1}{15}a^2x' = 0 \quad \ldots\ldots (13),$$

which, rejecting the solution $x' = 0$, corresponding to the obvious horizontality of the rod at its centre, gives $x' = \pm\frac{a}{\sqrt{15}}$ for the equal distances, in opposite directions, of the two points of greatest linear deflection of the mean segment from the centre of the rod.

The two extreme segments also have, of course, points of greatest linear deflection equidistant in opposite directions from the centre of the rod; but the cubic obtained from (3) and (10) for the determination of their distances from the centre, viz.,

$$(a-x)^3 - \tfrac{4}{5}a(a-x)^2 + \tfrac{2}{45}a^3 = 0 \quad \ldots\ldots (14),$$

hough having all its roots real, and one of them lying within the limits of the segments, admits only of an approximate solution in this case.

5783. (By S. Tebay, B.A.) — If m cards, consisting of s suits of $a_1, a_2, \ldots\ldots a_s$ cards respectively, be shuffled, find the probability that each suit consists of p partitions.

I. *Solution by the* Rev. W. A. Whitworth, M.A.

The first suit of a cards can be arranged into p groups in

$$\frac{a_1!\ a_1-1!}{a_1-p!\ p!\ p-1!}$$

ways. (*Choice and Chance*, Prop. xvii.) And so for the other suits. Hence all the suits can be arranged, into p groups each, in

$$\frac{(a_1!\ a_2!\ \ldots\ a_s!)\ .\ (a_1-1!\ a_2-1!\ \ldots\ a_s-1!)}{a_1-p!\ a_2-p!\ a_3-p!\ \ldots\ a_s-p!\ (p!\ p-1!)^s}$$

ways ($\equiv$ N ways, suppose). We have now to arrange our ps groups

(which are of s sorts, p of every sort) so that no two groups of the same sort may come together. If we take only two of the sorts, we can arrange the $2p$ things in $2 \,.\, p\,!\, p\,!$ ways. A third set of p groups can be introduced among these in $2p+1\,!+p+1\,!$ ways. A fourth set can then be introduced in $3p+1\,!+2p+1\,!$ ways, and so on. Finally the s^{th} set can be introduced in $sp-p+1\,!+sp-2p+1\,!$ ways. Hence all the ps groups can be arranged in $2\,.\,p\,!\,sp-p+1\,!\div(p+1)$ ways. Therefore the whole number of favourable arrangements of the m cards is $2N\,.\,p\,!\,sp-p+1\,!\div(p+1)$, and the required chance will be the ratio of this number to $m\,!$ the total number of possible arrangements. Hence the chance is

$$\frac{2\,.\,sp-p+1\,!\,.\,(a_1\,!\,a_2\,!\,\ldots\,a_s\,!)\,.\,(a_1-1\,!\,a_2-1\,!\,\ldots\,a_s-1\,!)}{(p+1)\,(p\,!)^{s-1}\,(p-1\,!)^s\,a_1-p\,!\,a_2-p\,!\,\ldots\,a_s-p\,!\,m\,!}.$$

II. *Solution by the* PROPOSER.

Let the equations $a_1+a_2+\ldots+a_p=a_1$, $\beta_1+\beta_2+\ldots+\beta_p=a_2$, $\gamma_1+\gamma_2+\ldots+\gamma_p=a_3$, &c., admit of α, β, γ, &c. solutions, and let one form of partition be represented by

$$a_1\,\beta_1\,\gamma_1\ldots\,a_2\,\beta_2\,\gamma_2\ldots\,a_p\,\beta_p\,\gamma_p\ldots.$$

This arrangement gives $(s\,!)^p$ variations, including repetitions. To find the number of repetitions, let letters of the same name be joined throughout the p groups. While these are unchanged, the other letters give $(s-1\,!)^2\,(s-2\,!)^{p-2}$ variations. Now, there are $p-1$ connections, and any two consecutive groups can form s connections, while the others run out in consecutive pairs to both extremes, giving $s-1$ connections at each step, or $(p-1)\,s\,(s-1)^{p-2}$ in all; consequently the number of repetitions

$$=(p-1)\,s\,(s-1)^{p-2}\,(s-1\,!)^2\,(s-2\,!)^{p-2}=(p-1)\,s\,(s-1\,!)^n.$$

Hence the number of partitions

$$=(s\,!)^p-(p-1)\,s\,(s-1)^p;$$

and, interchanging the limits, the chance required

$$=(p\,!)^s\left\{(s\,!)^p-(p-1)\,s\,(s-1\,!)^p\right\}\alpha\beta\gamma\ldots\div\frac{m\,!}{a_1\,!\,a_2\,!\,\ldots\,a_s\,!}.$$

Example. Let $p=2$, $s=3$, and take $a_1=3$, $a_2=4$, $a_3=5$; then $\alpha=1$, $\beta=2$, $\gamma=2$. Also $m=12$; and the chance $=$

$$(2\,!)^3\left\{(3\,!)^2-3\,.\,(2\,!)^2\right\}.4\div\frac{12\,!}{3\,!\,4\,!\,5\,!}=\frac{16}{1155}.$$

[In regard to the foregoing solution, Mr. WHITWORTH inquires: "Why does not Mr. TEBAY evaluate $\alpha\beta\gamma\ldots$ in his final result? We could they test the accuracy of his result by applying it to particular cases. But, if I understand the question aright, Mr. TEBAY's reasoning is quite unsound, and his result wrong. Consider the case when $p=1$, then the question is: to find the chance that all the cards of each suit are together.

My result comes to $\dfrac{s\,!\,a_1\,!\,a_2\,!\,\ldots\,a_s\,!}{m\,!}$, which is obviously right.

Mr. TEBAY's comes to $\dfrac{s-1\,!\,a_1\,!\,a_2\,!\,\ldots\,a_s\,!}{m\,!}$, which is obviously wrong."

To this Mr. TEBAY rejoins that, after a careful examination, he is con-

vinced that his solution is right, and that Mr. WHITWORTH and he only differ in the fact that he has partitioned *numbers* and Mr. WHITWORTH partitioned *things*; that, in short, Mr. WHITWORTH seems to consider each suit made up of different cards, as if marked 1, 2, 3, &c., while he has himself considered them as of one colour; namely, a_1 red, a_2 white, &c. Mr. TEBAY adds that there is a difference between the partitioning of things and the partitioning of numbers; and, for this reason, he considers that both solutions may stand as, from different points of view, equally correct.

Mr. WHITWORTH, having seen Mr. TEBAY's rejoinder, still adheres to his former assertion, and points out that the chance of the a_1 red cards being in p partitions after shuffling cannot be affected by the question whether these cards are *inter se* distinguishable or indistinguishable.]

5875. (By the Rev. Dr. HAUGHTON, F.R.S.)—Show that, within the limits of NEWTON's construction for the disturbing force, the *equilibrium tidal surface* (statical problem) of an equatorial canal of constant width and depth (without rotation), attracted by the moon always in the equator, is

$$\log(\rho) = \alpha \cos 2\phi + \beta.$$

Solution by J. J. WALKER, M.A.

In this case I resolve the forces acting on the particle of water P along PT (ρ), T being the Earth's centre, and PY parallel to the line joining centres of Earth and Moon. The disturbing forces in these directions are $\frac{k\rho}{R}$ and $\frac{3k\rho\cos\phi}{R}$ respectively, within the specified limits; and adding to the former $\frac{kg'a^2}{R\rho^2}$ for Earth's attraction, where $g' = \frac{gR}{k}$, the condition that the forces on P shall have a resultant along PN the normal to the free surface, meeting the line of centres in N, is

$$\rho + \frac{g'a^2}{\rho^2} : 3\rho\cos\phi = PT : TN = d(\rho\cos\phi) : d\rho, \text{ [see Quest. 5964,]}$$

or
$$\left(\rho + \frac{g'a^2}{\rho^2}\right) d\rho = 3\rho\cos\phi\, d(\rho\cos\phi),$$

whence
$$\rho^2 - \frac{2g'a^2}{\rho} + C = 3\rho^2\cos^2\phi.$$

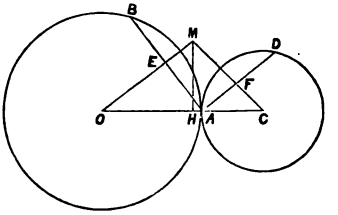
If, in the expressions for the disturbing forces and Earth's attraction, we take $\rho = a$, and $d(\rho\cos\phi) = \rho d\cos\phi$, then the equation to be integrated will be
$$\frac{a+g'}{a\rho}\, d\rho = 3\cos\phi\, d(\cos\phi),$$

whence
$$\frac{a+g'}{a}\log\rho = \tfrac{3}{2}\cos^2\phi + C,$$

which is of the form proposed in the Question, but the one above is obviously more exact.

5855. (By Professor DARBOUX.)—Par le point de contact A de deux circonférences données, on mène deux cordes AB et AD qui soient dans un rapport donné ; et des centres O, C on abaisse des perpendiculaires sur ces cordes. On demande le lieu géométrique du point de rencontre M de ces perpendiculaires.

Solution by Professor COCHEZ ; G. TURRIFF, M.A. ; *and others.*



Du point M abaissons la perpendiculaire MH sur la ligne des centres. Les triangles semblables MOH et AOE, MCH et AFC, donnent

$$\frac{MO}{MH} = \frac{OA}{AE}, \quad \frac{CM}{MH} = \frac{CA}{FA}.$$

Divisant membre à membre ces deux relations,

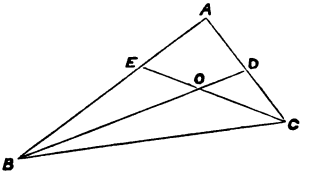
$$\frac{OM}{CM} = \frac{OA \cdot FA}{CA \cdot EA} = \text{Constante};$$

puis que $\frac{FA}{EA}$ est constant par hypothèse.

Le lieu est donc une circonférence décrite sur la droite qui joint les points conjugués harmoniques partageant OC dans le rapport donné.

5866. (By H. N. CAPEL, LL.B)—Given the two lines bisecting the acute angles of a right-angled triangle ; determine the triangle.

Solution by MORGAN BRIERLEY ; EDWYN ANTHONY, M.A. ; *and others.*



From any point O, draw OB equal to one of the given bisectors ; make $\angle BOC = \frac{3}{4}\pi$; take OC = the other given bisector ; join BC, and make $\angle ABO = OBC$; and $\angle OCA = OCB$; then ABC is the required triangle. For, from the triangle BOC, we have

$$\frac{3\pi}{4} + \tfrac{1}{2}(B + C) = \pi ;$$

therefore $B + C = \frac{1}{2}\pi$, therefore $A = \frac{1}{2}\pi$.

[Our correspondents consider the bisectors in the question to terminate at their point O of intersection. These bisectors may, however, with equal propriety, be taken as terminated at the points D, E in the opposite sides of the triangle ; and in this form the problem has been reproposed for solution as Question 5885.]

5853. (By Professor CROFTON, F.R.S.)—Explain the following paradox:—One bag (A) contains 5 cheques, whereof 2 are known to be over £5 and 3 under. Another bag (B) contains 7 cheques, 4 over £5 and 3 under. Find the chance, if a cheque be drawn from A, that it exceeds in amount one drawn from B. There are 35 cases, all equally probable; 6 evidently favourable, 12 unfavourable, and 17 doubtful. In these 17 cases, as all we know is that the 2 cheques drawn are both over £5, or both under, it is an even chance that one exceeds the other. Hence the chance required, that the cheque from A exceeds that from B, is $\frac{6}{35} + \frac{1}{2}(\frac{17}{35}) = \frac{29}{70}$. But the chance that the cheque from A exceeds £5 is $\frac{2}{5} = \frac{28}{70}$, so that there is a *less* chance that the cheque drawn from A exceeds £5 than that it exceeds an (unknown) quantity which is likely to be greater than £5; the odds being 4 to 3. *A priori*, one would say it must be the other way.

Solution by W. J. MACDONALD, M.A.

Though the chance that a cheque from A exceeds one from B is $\frac{29}{70}$, yet the chance that the latter exceeds £5 is only $\frac{4}{7}$; therefore the chance that a cheque from A exceeds a quantity greater than £5 is only $\frac{29}{70} \times \frac{4}{7} = \frac{58}{245}$.

5831. (By Professor MINCHIN, M.A.)—If S and E be a conic and its evolute, and if N be the conic whose intersections with S are the feet of the normals drawn to S from a point P: prove that (1) the perpendicular from P on its polar with respect to S is a tangent to N at P; (2) the centre of gravity of the feet of the four normals is the middle point of the line joining the centres of S and N; (3) the locus of points whence normals drawn to S form a harmonic pencil is $E + 27a^2b^2c^4x^2y^2 = 0$; (4) E being $\equiv C^2 + 27a^2b^2c^4x^2y^2$, if P be any point on the conic C, the locus of the centre of N is S.

Solution by R. GRAHAM, M.A.; S. JOHNSTON, M.A.; *and others.*

1. Let $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, $N \equiv c^2xy + b^2y'x - a^2x'y = 0$,

$$E \equiv (a^2x^2 + b^2y^2 - c^4)^2 + 27a^2b^2c^4x^2y^2 = C^2 + 27a^2b^2c^4x^2y^2 = 0.$$

The conic N manifestly passes through $x'y'$, the common point of the four normals. The equation of the tangent to N at $x'y'$ is

$$\frac{xy'}{b^2} - \frac{yx'}{a^2} = \frac{c^2x'y'}{a^2b^2}, \text{ which is at right angles to } \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

the polar of $x'y'$ with respect to S.

2. If ϕ be the eccentric angle of the foot of one of the normals through $x'y'$, then

$$\frac{ax'}{\cos\phi} - \frac{by'}{\sin\phi} = c^2,$$

which can be written in either of the forms

$$c^4\cos^4\phi - 2ac^2x'\cos^3\phi + C'\cos^2\phi + 2ac^2x'\cos\phi - a^2x'^2 = 0 \ldots\ldots (1),$$

or $$c^4\sin^4\phi + 2bc^2y'\sin^3\phi + C'\sin^2\phi - 2bc^2y'\sin\phi - b^2y'^2 = 0.$$

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity of the feet of the four normals through (x', y'), we have from these last equations

$$\bar{x} = \frac{a^2x'}{2c^2} \text{ and } \bar{y} = -\frac{b^2y'}{2c^2}.$$

The coordinates of the centre of N are $x = \frac{a^2x'}{c^2}$, $y = -\frac{b^2y'}{c^2}$;

hence the coordinates of the middle point of the line joining the centres of S and N are equal to $\bar{x}$ and $\bar{y}$ respectively.

3. The intercept on the axis of x by one of the normals is $\frac{c^2}{a}\cos\phi$; if we multiply each root of equation (1) by $\frac{c^2}{a}$, we shall have an equation whose roots are the intercepts made by the four normals through (xy); viz., $a^2\lambda^4 - 2a^2x\lambda^3 + C\lambda^2 + 2c^2x\lambda - c^4x = 0$.

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the roots of this biquadratic; then, since the normals form a harmonic pencil, $\frac{\lambda_3-\lambda_4}{\lambda_2-\lambda_3} = \frac{\lambda_1-\lambda_4}{\lambda_1-\lambda_2}$,

or $\frac{1}{2}(\lambda_1\lambda_3 + \lambda_2\lambda_4) = \frac{C}{6a^2}$ = a root of SIMPSON's reducing cubic;

hence we find $\frac{C^3}{27a^6} + \frac{2c^4b^2x^2y^2}{a^4} = 0$ or $E + 27a^2b^2c^4x^2y^2 = 0$.

4. By Art. 2, $ax' = c^2\frac{x}{a}$, and $by' = -c^2\frac{y}{b}$; where x, y are the coordinates of the centre of N. Substituting in $C=0$, since $x'y'$, by hypothesis, lies on C, we find $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ for the locus of the centre of N.

5906. (By Professor MATZ, M.A.)—In a given ellipse, determine the average length of all the radius-vectors that can be drawn from one of the foci through the arc intercepted by the focal ordinates.

Solution by R. E. RILEY, B.A.; A. W. SCOTT, M.A.; *and others.*

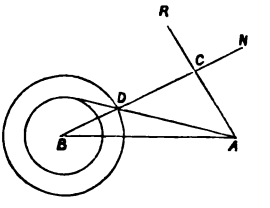
The polar equation of the ellipse being $\rho = \frac{a(1-e^2)}{1-e\cos\theta}$, the average required is, putting $\theta_1 = \cos^{-1}\left(\frac{2e}{1+e^2}\right)$,

$$\frac{\int\rho d\theta}{\int d\theta} = \frac{a(1-e^2)}{\frac{1}{2}\pi-\theta_1}\int_{\theta_1}^{\frac{1}{2}\pi}\frac{d\theta}{1-e\cos\theta}$$

$$= \frac{2a(1-e^2)^{\frac{1}{2}}}{\frac{1}{2}\pi-\theta_1}\left\{\tan^{-1}\left[\tan\tfrac{1}{2}\theta\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\right]\right\}_{\theta_1}^{\frac{1}{2}\pi} = \frac{b\sin^{-1}e}{\tan^{-1}e}.$$

5798. (By Professor Cochez.)—Construire un triangle rectangle, connaissant les sommes de chacun des côtés avec la perpendiculaire abaissée du sommet de l'angle droit sur l'hypotenuse.

Solution by R. E. Riley, B.A.; J. O. Regan; *and others.*



We are given $a+p$, $b+p$, and the hypotenuse AB. From centre B, and with radius $a-b$, draw a circle; and from the centre B, with radius whose square is $\frac{1}{2}(a-b)$, draw another circle. From A draw a tangent to the latter circle, cutting the former in D. Join BD and produce it to N, making $BN = a+p$. Draw AC at right angles to BN, and produce it to R, making CR = CN. Then, if $AR = b+p$, ABC is the triangle required. For $\angle CDA = 45°$, therefore CA = CD;

$DN = a+p-(a-b) = b+p$; and DN = AR, therefore $AR = b+p$; and ABC is the required triangle.

5896. (By E. W. Symons.)—Prove that the volume of a tetrahedron, whose vertices are any vertex of a given tetrahedron, and the centroids of the faces meeting in that vertex, is $\frac{8}{27}$ of the volume of the given tetrahedron.

Solution by Rev. D. Thomas, M.A.; R. Tucker, M.A.; *and others.*

Let O be a vertex, ABC the opposite face, and D, E, E the middle points of BC, CA, AB. Then the volume of the tetrahedron in the question is clearly $\frac{8}{27}$ ODEF; and

vol. ODEF : vol. OABC = $\triangle$DEF : $\triangle$ABC = 1 : 4,

therefore given volume = $\frac{1}{4}$ of $\frac{8}{27}$ of OABC = $\frac{2}{27}$OABC.

5862. (By Edwyn Anthony, M.A.)—Prove that $n^{13}-n$ is always divisible by 2730, whatever whole number n may be.

Solution by R. Tucker, M.A.; W. J. Macdonald, M.A.; *and others.*

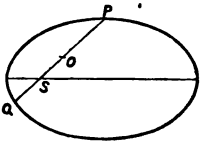
$$N \equiv n^{13}-n = 2730 = n(n^{12}-1).$$

By Fermat's theorem, since $n^{12}-1 = (n^6-1)(n^6+1)$, $n^{12}-1$ is divisible by 13 and by 7; also, since $n^{12}-1 = (n^4-1)(n^8+n^2+1)$, it is divisible by 5; and, as it is divisible by $(n-1)n(n+1)$, it is also divisible by 6; therefore N is divisible by $13.7.5.6 = 2730$.

5887. (By Edwyn Anthony, M.A.)—If O be the middle point of a focal chord PSQ of an ellipse, and SP equals its semi-major-axis; show that the ratio of OS to OP equals the square of the eccentricity.

Solution by C. F. D'Arcy, M.A.; A. W. Scott, M.A.; *and others.*

The point P will be at an extremity of the minor axis. If L be semi-latus-rectum, we have



$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{L};$$

hence $SQ = \frac{b^2}{a(1+e^2)}$, and $PQ = \frac{2a}{1+e^2}$;

therefore $OS = SP - \frac{1}{2}PQ = \frac{ae^2}{1+e^2} = e^2 . OP.$

5828. (By Professor Darboux.)—On coupe une pyramide triangulaire SABC par un plan parallèle à la base; ce plan rencontre les arêtes latérales SA, SB, SC en A′, B′, C′; on mène ensuite les plans CA′B′, AB′C′, BC′A′. Soit P leur point commun. Déterminer le lieu décrit par le point P lorsque le plan A′BC′ se déplace en demeurant parallèle à la base.

Solution by R. Graham, M.A.; T. R. Terry, B.A.; *and others.*

Let BC′ and B′C intersect in O_2 then SO produced will evidently bisect BC. A′O is the line of intersection of the planes BC′A′ and CA′B′, which lies in the plane through AS bisecting the opposite edge BC. Hence the point whose locus we seek lies on the line joining S with the centre of gravity of the face ABC.

5870. (By J. L. Mackenzie, B.A.)—The evolute of a non-singular cubic has in general twenty-seven cusps. If P be a point on the cubic corresponding to any of these cusps, shew that the remaining points in which the circle of curvature at P cuts the cubic are collinear with the *second tangential* of P. [The term "second tangential" is employed by Salmon in his *Higher Plane Curves*, 2nd ed., Art. 155. The tangent at any point P on a cubic meets the curve again in a point which is called the *tangential* of P; and the tangent at the latter point meets the curve again in a point which is the *second tangential* of P.]

Solution by the PROPOSER.

When there is a cusp on the evolute of the cubic, three consecutive normals to the cubic pass through a point, and therefore four consecutive points on the cubic lie on the circle of curvature at P. This circle cuts the cubic again in two points A, B; and the straight line AB cuts the cubic again in R, the point-residual of the four consecutive points at P. Again, the tangent at P, taken twice, passes through the four consecutive points at P, and cuts the cubic again in two consecutive points at Q, the first tangential of P; and the tangent at Q cuts the cubic again in R′, the second tangential of P. And, by the theory of residuation, this point R′ is the same as R, the point-residual of the four consecutive points at P.

5802. (By T. R. TERRY, M.A.)—If O be a fixed point in a rigid body, and the direction cosines of the lines OA, OB, OC be represented by (l, m, n), (l', m', n'), (λ, μ, ν), respectively; show that a rotation of the body round the axis OA through an angle 2θ, followed by a rotation round the axis OB through an $\angle 2\theta'$, is equivalent to a rotation round the axis OC through an angle 2ϕ, where

$$\frac{\cos\phi}{\cos\theta'\cos\theta - \sin\theta'\sin\theta\,(ll' + mm' + nn')} = \frac{\lambda\sin\phi}{X} = \frac{\mu\sin\phi}{Y} = \frac{\nu\sin\phi}{Z},$$

where

$X = l\cos\theta'\sin\theta + l'\cos\theta\sin\theta' + \sin\theta'\sin\theta\,(m'n - mn')$, $Y =$ &c., $Z =$ &c.

Solution by J. J. WALKER, M.A.

These expressions may be found substantially in Hamilton's earlier papers on Quaternions. The first application of his new method, which he communicated to the R. I. A. (*Proc.*, Vol. III., 1834), was to the composition of translations and rotations; and he gave for the rotation of a vector β round a fixed unit vector α, through an angle 2θ, the beautiful formula, $\beta' = (\cos\theta + \alpha\sin\theta)\,\beta\,(\cos\theta - \alpha\sin\theta)$. Similarly, if by a second rotation of β' about α' through an $\angle 2\theta'$, it becomes β'', then

$$\beta'' = (\cos\theta' + \alpha'\sin\theta')\,\beta'\,(\cos\theta' - \alpha'\sin\theta')$$
$$= (\cos\theta' + \alpha'\sin\theta')\,(\cos\theta + \alpha\sin\theta)\,\beta\,(\cos\theta - \alpha\sin\theta)\,(\cos\theta' - \alpha'\sin\theta').$$

By a single rotation through an angle 2ϕ, round an axis α_1, β may be made to coincide with β'' if $\cos\phi$ is identified with the scalar part of the product $(\cos\theta' + \alpha'\sin\theta')\,(\cos\theta + \alpha\sin\theta)$, which by actual multiplication is $\cos\theta\cos\theta' - \sin\theta\sin\theta'\,(ll' + mm' + nn')$, since, i, j, k being unit vectors in the directions of the axes of coordinates,

$$\alpha = li + mj + nk, \quad \alpha' = l'i + m'j + n'k;$$

while, by the identification of the vector part of $\cos\phi + \alpha_1\sin\phi$, viz., $(\lambda i + \mu j + \nu k)\sin\phi$, with that of the same product,

$$\lambda\sin\phi = X, \quad \mu\sin\phi = Y, \quad \nu\sin\phi = Z.$$

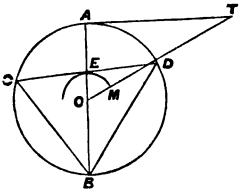
Thus the fractions, proposed to be proved equal, are actually each equal to unity. From the formulæ thus found, Hamilton deduced his construction of the resultant axis as the radius drawn to the third corner of a

spherical triangle, the other two corners of which are determined by the two component axes, the magnitudes of the angles at these corners being θ, θ' respectively; while ϕ is equal to the supplement of the angle at the first. [See also TAIT's "*Quaternions*," 2nd ed., Arts. 351, 352.]

5768. (By Professor DUPAIN.)—Déterminer sur un diamètre AB d'une sphère de rayon R, un point tel que si l'on mène par ce point un plan perpendiculaire à ce diamètre, la surface de la zône sphérique limitée par le plan et contenant le point A, soit équivalente à la surface latérale du cône qui a pour base le cercle d'intersection de la sphère et du plan et pour sommet le point B. Celà étant, calculer le rapport du volume de ce cône au volume de la sphère.

Solution by Professor COCHEZ; H. L. ORCHARD, B.A.; *and others.*

Soit E le point cherché. Posons $OE = x$, on doit avoir



$$2\pi R \cdot AE = \pi CE \cdot BC,$$

ou $$2R \cdot AE = CE \cdot BC \ldots\ldots\ldots\ldots (1),$$

R étant le rayon de la sphère. Mais

$$AE = R - x, \quad CE = (R^2 - x^2)^{\frac{1}{2}},$$

$$CB = [2R(R+x)]^{\frac{1}{2}}.$$

Portant ces valeurs dans (1), on

$$[2R(R-x)]^{\frac{1}{2}} \{[2R(R-x)]^{\frac{1}{2}} - (R+x)\} = 0,$$

égalité satisfaite pour $x = R$; dans ce cas les deux surfaces sont nulles, et pour $[2R(R-x)]^{\frac{1}{2}} = R + x$, qui donne $x^2 + 4Rx - R^2 = 0$,

d'où $$x' = R(\sqrt{y} - z), \quad x'' = -R(z + \sqrt{y});$$

cette seconde valeur x'' ne saurait convenir puisque sa valeur absolue est plus grande que R. Il reste donc comme solutions

$$x = 0, \quad x = R(\sqrt{y} - z).$$

Le rapport K des volumes est $K = \frac{\frac{1}{3}\pi (R^2 - x^2)(R+x)}{\frac{4}{3}\pi R^3} = \frac{(R^2 - x^2)(R+x)}{4R^3}$

ou, en remplaçant x par sa valeur, $K = 7 - 3\sqrt{y}$.

La valeur de x se construit facilement. Pour cela on mène en A une tangente $AT = 2R$; on joint OT, on a $OT = R\sqrt{y}$, on prend $TM = 2R$ et on décrit de O comme centre avec OM pour rayon un cercle qui coupe OA en E, E est le point qui répond à la question.

5851. (By Prof. TOWNSEND, F.R.S.)—A material particle being supposed in equilibrium, in a free space, under the action, attractive or repul-

sive, of a system of masses distributed in any manner in the space; show, generally, that its equilibrium, when not absolutely neutral, is always perfectly stable for attractive, and perfectly unstable for repulsive, action, or is, on the contrary, always intermediate, according as the law of the action is a direct or an inverse power of the distance.

I. *Solution by* W. J. Curran Sharp, M.A.

If m_1, m_2, m_3. &c. be the masses; a, b, c, &c. their coordinates; $r_1 r_2$, &c. the distances from (xyz), the attracted particle; so that

$$r_1^2 = (a_1-x)^2+(b_1-y)^2+(c_1-z)^2;$$

the attractions parallel to the axes are for a force varying as the n^{th} power of

$$\mp\Sigma r_1^{n-1}(a_1-x) = 0 \equiv \text{X parallel to axis of } x,$$
$$\mp\Sigma r_1^{n-1}(b_1-y) = 0 \equiv \text{Y} \quad ,, \quad y,$$
$$\mp\Sigma r_1^{n-1}(c_1-z) = 0 \equiv \text{Z} \quad ,, \quad z,$$

the upper sign corresponding to an attractive and the lower to a repulsive force.

Then the equilibrium will be stable, unstable, or intermediate, according as $\int(X dx + Y dy + Z dz)$ is a maximum, a minimum, or neither; and this integral becomes $\pm\frac{\Sigma r_1^n}{n}$, and using the notation of Williamson's *Diff. Calc.*, p. 183,

$$A = \pm\Sigma\{r_1^{n-1}+(n-1)r_1^{n-2}(a_1-x)^2, \&c.\},$$
$$H = \pm\Sigma(n-1)r_1^{n-2}(a_1-x)(b_1-y), \&c.;$$

and the expression $A\alpha^2+$ &c., reduces to

$$\pm\Sigma r_1^{n-1}(\alpha^2+\beta^2+\gamma^2) \pm (n-1)\Sigma r_1^{n-2}\{\overline{a_1-x}\,\alpha+\overline{b_1-y}\,\beta+\overline{c_1-z}\,\gamma\}^2,$$

which is essentially positive for an attractive and negative for a repulsive force if n be positive, and is intermediate for negative values of n.

II. *Solution by the* Proposer.

To prove this, it is only necessary to show, in accordance with a well-known general principle in dynamics, the application of which to the case in question is obvious, that, special circumstances excepted, the potential V of the system is always a minimum in the former case, and only stationary in the latter case; which may be readily done as follows:—

Conceiving the particle to receive a small displacement $\xi\eta\zeta$ from its position of equilibrium xyz under the action of the system; it is to be shown that the function, representing accurately in sign and approximately in magnitude the consequent change in V; viz.,

$$\left(\frac{d^2V}{dx^2}\xi^2+\frac{d^2V}{dy^2}\eta^2+\frac{d^2V}{dz^2}\zeta^2+2\frac{d^2V}{dydz}\eta\zeta+2\frac{d^2V}{dzdx}\zeta\xi+2\frac{d^2V}{dxdy}\xi\eta\right)$$

preserves a common sign and is positive for all values of $\xi\eta\zeta$ in the former case, and does not in the latter case.

Since, generally, for any power n of the distance,

$$(n+1)V = \Sigma(m.r^{n+1}) = \Sigma\left[m\{(x-a)^2+(y-b)^2+(z-c)^2\}^{\frac{1}{2}(n+1)}\right]$$

where abc, $a'b'c'$, &c. are the coordinates of m, m', &c. in the space; therefore, performing the several differentiations, and adding together and arranging the several results, we have, for the function in question, the value

$$\Sigma\left[mr^{n-3}\left\{[(x-a)^2+(y-b)^2+(z-c)^2]\,[\xi^2+\eta^2+\zeta^2]\right.\right.$$
$$\left.\left.+[n-1]\,[(x-a)\,\xi+(y-b)\,\eta+(z-c)\,\zeta]^2\right\}\right];$$

or, as it may be written more shortly

$$\Sigma\left[mr^{n-3}\left\{r^2\rho^2+(n-1)\,r^2\rho^2\cos^2\theta\right\}\right],$$

where ρ is the vector of the displacement, and θ the angle between the directions of r and ρ.

But this manifestly preserves a common sign, and is positive, for all values of ρ and θ, when n is positive, and does not preserve a common sign at all, as ρ and θ vary, when n is negative; and therefore, &c., as regards the property.

The quadric cone, real or imaginary, whose equation, in rectangular coordinates to the position of equilibrium xyz as origin, is

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0,$$

where
$$A=\Sigma\left[mr^{n-3}\left\{r^2+(n-1)\,(x-a)^2\right\}\right],\ \&c.,$$
$$F=\Sigma\left[mr^{n-3}\left\{(n-1)\,(y-b)\,(z-c)\right\}\right],\ \&c.,$$

evidently divides, in all cases, the space about the point into the two different regions, for which the changes in V resulting from the displacement of the particle have opposite signs in the above.

The vector ρ of the displacement being supposed constant; the squared reciprocal of any radius of the central quadric to the same origin and axes,

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=k^2,$$

where k is any constant, represents as evidently, in all cases, in magnitude and sign, the change in V resulting from a displacement of the particle in the direction of the radius.

2495. (By W. S. B. Woolhouse, F.R.A.S.) — Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude, such that, if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum.

Solution by the Proposer.

This problem was originally proposed by me as an exercise upon the practical application of the very general theorem established in Question 2471 (*Reprint*, Vol. VIII., page 100). It will here be convenient in the first place to reproduce the enunciation of the theorem, which is as follows:—

THEOREM.—Let a given surface having a convex boundary of *any form whatever*, be referred to its centre of gravity and the principal axes of rotation situated in its plane; and, corresponding to an abscissa x, let y, y' be the respective distances of the boundary above and below the axis; then, if h, k denote the radii of gyration round the axes, and M the total area, we shall have

$$h^2 = \int x^2\,dx\,\frac{y+y'}{M}, \quad k^2 = \tfrac{1}{3}\int dx\,\frac{y^3+y'^3}{M}.$$

Let also $\quad A = \int x^2\,dx\,\frac{y^3+y'^3}{M}, \quad B = \int x\,dx\,\frac{y\int y^2dx + y'\int y'^2dx}{M},$

$$C = \tfrac{1}{4}A + 3B;$$ then—

1. If *three* points be taken at random, on the given surface, the average area of the triangle connecting them, in parts of the total area, is $(\Delta) = \frac{h^2k^2+C}{M^2}$; and the average square of the area, expressed in parts of the square of the total area, is $(\Delta^2) = \frac{3}{4}\cdot\frac{h^2k^2}{M^2}$.

2. If *four* points be taken at random, the probability that the quadrilateral shall be reentrant $= 4\,(\Delta)$.

3. If *five* points be taken at random on the surface, the probability of

$$\begin{aligned} &\text{a convex pentagon} &&= 1 - 10\,(\Delta) + 10\,(\Delta^2),\\ &\text{one reentrant point} &&= \quad 10\,(\Delta) - 20\,(\Delta^2),\\ &\text{two reentrant points} &&= \qquad\quad 10\,(\Delta^2). \end{aligned}$$

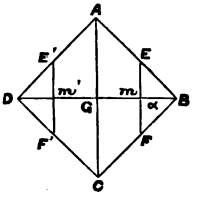
NOTE.—When the axis of x is diametral, then

$$h^2 = \frac{2}{M}\int yx^2\,dx, \quad k^2 = \frac{2}{3M}\int y^3\,dx,$$

$$A = \frac{2}{M}\int y^3x^2\,dx, \quad B = \frac{2}{M}\int yx\,dx\int y^2\,dx.$$

And in the formula for B, since $\int yx\,dx\,.\,c = 0$, the inner integral may be estimated from any epoch. If the axis of y be also diametral, the calculation of the integrals may be further simplified by taking them for only one quadrant and doubling the results just stated.

We proceed to apply the foregoing theorem to the special question under consideration. The probability of five random points determining a convex pentagon on the particular surface proposed will be of precisely the same value if the diagram be supposed to be first projected orthogonally upon any plane.



Assume the relative position of the plane to be such that the rectangle is projected into the square ABCD; then the lines of section, EF, E'F' in which the corners are cut off will still be parallel to the diagonal AC, and they will be bisected in m, m' by the other diagonal BD. The remaining surface in which the five random points are supposed to be taken is EAE'F'CF, and AC and DB are evidently the principal axes to this surface. We therefore take them as the axes of coordinates. As the probability in the question is dependent on relative magnitudes only, we may obviously assume AG = GB = 1; $Em = mB = \alpha$; and x, y the coordinates of any point in the boundary of

surface. The form of the surface being symmetrical with respect to both axes, we shall avail ourselves of the simplification pointed out in the latter part of the note to the theorem, and consider only the quadrant AGmE, in which the limits are evidently $x = 0$ to $1-\alpha$, and $y = 1-x$ throughout. Hence the following integrations :—

$$\int yx^2\,dx = \int_0^{1-\alpha}(1-x)\,x^2\,dx = \frac{(1-\alpha)^3}{3} - \frac{(1-\alpha)^4}{4} = (1-\alpha)^3\frac{1+3\alpha}{12} \quad \ldots\ldots(1),$$

$$\int y^3\,dx = \int_0^{1-\alpha}(1-x)^3\,dx = \frac{1-\alpha^4}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\int y^3x^2\,dx = \int_0^{1-\alpha}(1-x)^3x^2\,dx = \frac{(1-\alpha)^3}{3} - \frac{3(1-\alpha)^4}{4} + \frac{3(1-\alpha)^5}{5} - \frac{(1-\alpha)^6}{6}$$

$$= (1-\alpha)^3\frac{1+3\alpha+6\alpha^2+10\alpha^3}{60} \quad \ldots\ldots\ldots\ldots\ldots(3),$$

$$yx\,dx\int y^2\,dx = \int_0^{1-\alpha}(1-x)\,x\,dx\int(1-x)^2\,dx$$

$$= \int_0^{1-\alpha}(1-x)\,x\,dx\left(x-x^2+\frac{x^3}{3}\right) = (1-\alpha)^3\frac{4+12\alpha+9\alpha^2+5\alpha^3}{90} \ldots\ldots(4).$$

Also $$\tfrac{1}{4}\mathrm{M} = (1-\alpha)\,\alpha + \tfrac{1}{2}(1-\alpha)^2 = \tfrac{1}{2}(1-\alpha)(1+\alpha) \ldots\ldots\ldots\ldots(5).$$

Therefore, by the theorem, we have

$$h^2 = \frac{(1)}{(5)} = \frac{(1-\alpha)^2}{6}\cdot\frac{1+3\alpha}{1+\alpha}, \quad k^2 = \tfrac{1}{3}\frac{(2)}{(5)} = \frac{1+\alpha^2}{6};$$

$$\mathrm{A} = \frac{(3)}{(5)} = \frac{(1-\alpha)^2}{30}\cdot\frac{1+3\alpha+6\alpha^2+10\alpha^3}{1+\alpha},$$

$$\mathrm{B} = \frac{(4)}{(5)} = \frac{(1-\alpha)^2}{45}\cdot\frac{4+12\alpha+9\alpha^2+5\alpha^3}{1+\alpha},$$

$$\mathrm{C} = \tfrac{1}{3}\mathrm{A} + 3\mathrm{B} = \frac{(1-\alpha)^2}{18}\cdot\frac{5+15\alpha+12\alpha^2+8\alpha^3}{1+\alpha},$$

$$h^2k^2 = \frac{(1-\alpha)^2}{36}\cdot\frac{1+3\alpha+\alpha^2+3\alpha^3}{1+\alpha},$$

$$\mathrm{C}+h^2k^2 = \frac{(1-\alpha)^2}{36}\cdot\frac{11+33\alpha+25\alpha^2+19\alpha^3}{1+\alpha};$$

$$(\Delta) = \frac{\mathrm{C}+h^2k^2}{\mathrm{M}^2} = \frac{11+33\alpha+25\alpha^2+19\alpha^3}{144(1+\alpha)^3} = \frac{11}{144} - \frac{\alpha^2(1-\alpha)}{18(1+\alpha)^3},$$

$$(\Delta^2) = \frac{3}{2}\frac{h^2k^2}{\mathrm{M}^2} = \frac{1+3\alpha+\alpha^2+3\alpha^3}{96(1+\alpha)^3} = \frac{1}{96} - \frac{\alpha^2(1-\alpha)}{48(1+\alpha)^3}.$$

$$(\Delta)-(\Delta^2) = \frac{19}{288} - \frac{5\alpha^2(1-\alpha)}{144(1+\alpha)^3}, \quad (\Delta)-2(\Delta^2) = \frac{1}{18} - \frac{\alpha^2(1-\alpha)}{72(1+\alpha)^3};$$

therefore $$1-10(\Delta)+10(\Delta^2) = \frac{49}{144} + \frac{25\alpha^2(1-\alpha)}{72(1+\alpha)^3},$$

$$10(\Delta)-20(\Delta^2) = \frac{5}{9} - \frac{5\alpha^2(1-\alpha)}{36(1+\alpha)^3}, \quad 10(\Delta^2) = \frac{5}{48} - \frac{5\alpha^2(1-\alpha)}{24(1+\alpha)^3}.$$

Hence, if we put $$\mu = \frac{\alpha^2(1-\alpha)}{72(1+\alpha)^3},$$

the values of the several probabilities come out as follows:—

Reentrant quadrilateral $\frac{1}{18} - 16\mu$

Convex quadrilateral $\frac{5}{18} + 16\mu$ One reentrant point $\frac{5}{8} - 10\mu$

Convex pentagon $\frac{49}{144} + 25\mu$ Two reentrant points $\frac{5}{48} - 15\mu$

The probability of a convex pentagon is a maximum when the function μ is a maximum; and, putting $\frac{d\mu}{da} = 0$, we find $a = \frac{1}{2}$, giving also $\mu = \frac{1}{1944}$.

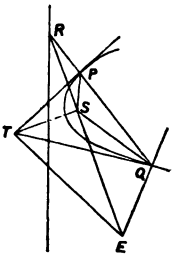
Thus it will appear that the lines of section suggested in the question will bisect the sides of the given rectangle; and, in such case, it is remarkable that the remaining surface will admit of being orthogonally projected from or into a regular hexagon. Indeed it may be concluded generally that, if five random points are taken on any polygonal surface, the probability that they shall be the apices of a convex pentagon will be a maximum when the surface is either a regular polygon or a projection of a regular polygon.

5913. (By C. Taylor, M.A.)—Prove that (1) a normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord; and (2) the polar of the middle point of the chord meets the focal vector to its point of concourse with the directrix upon the normal at its further extremity.

I. *Solution by* G. Turriff, M.A.; R. Knowles, L.C.P.; *and others.*

1. Let S be the focus, and join ST, SR. Then the foot R of the chord PQ, and the pole T, subtend a right angle at S (Taylor's *Geometry of Conics*, Art. 112). Hence a circle will go round RPST. Therefore $\angle$ TRS = TPS = STQ; hence TQ touches the circle, and is therefore at right angles to RT.

2. Let RS produced meet the normal at Q in E. Then, since a circle will go round STEQ, $\angle$ ETQ = ESQ = RSP (since SR bisects the supplement of PSQ) = RTP (in same segment of RPST) = TQP, by (1); hence ET is parallel to PQ.



II. *Solution by* C. F. D'Arcy, M.A.; D. Edwardes; *and others.*

1. Let (ξ, η) be the pole of the chord of $y^2 - 4ax = 0$, normal at (x', y'); then $x' = -(2a + \xi)$, and $ay' = (2a + \xi)\eta$, which gives $(2a + \xi)\eta^2 + 4a^3 = 0$ as the locus of the poles of normal chords (Quest. 5088); and if (α, β) is its point of concourse with the directrix

$$\xi - \alpha = a + \xi, \quad \eta - \beta = (a + \xi)\xi\eta + 2a^2,$$

whence

$$\eta - \beta : \xi - \alpha = \xi\eta : 2a^2.$$

But the equation to the second normal is

$$y + \frac{\xi\eta}{2a^2}\left(-x + 2a + \frac{\xi^2}{2a+\xi}\right) = 0 \quad \text{.....(i.)}$$

2. The polar of the middle point of the normal chord is

$$2a^2(y-\eta) + (2a+\xi)\eta(x-\xi) = 0 \quad \text{.....(ii.),}$$

and the focal vector to $\alpha\beta$ is

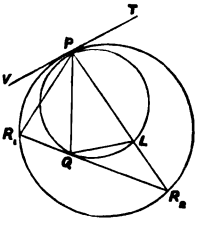
$$4a^3y + (a-\xi)(za+\xi)(x-a) = 0 \quad \text{.....(iii.)}$$

Now (i.) × (2a + ξ) + (ii.) × (2a − ξ) − (iii.) × 2 vanishes identically.

5380. (By W. Gallatly, B.A.)—If a circle A touches internally another circle B at P, and a tangent to A at the point Q intersect B in R_1, R_2, prove that $\angle R_1PQ = \angle R_2PQ$.

[This reciprocates into the following:—S is the common focus of two conics having contact of the second order at P_1; if from a point Q on the outer conic, two tangents QR_1, QR_2 be drawn to the inner conic intersecting the common tangent at P in the points R_1, R_2, and if the tangent at Q to the outer conic intersect the tangent at P in T; then TR_1, TR_2 subtend equal angles at the focus.]

Solution by W. J. Macdonald, M.A.; Donald Macalister, B.A., B.Sc.; *and others.*



At P draw the common tangent VPT, and join QL; then $\angle PR_1Q = TPL = PQL$,

and $\angle PQR_1 = PLQ$,

therefore $\angle R_1PQ = QPL$.

[This is a better proof of the theorem than the one given on p. 26 of Vol. XXIX. of the *Reprint.*]

5786. (By A. Martin, M.A.)—A square hole is cut through the centre of a sphere of radius r; show that the average volume removed is

$$\tfrac{1}{12}(23\sqrt{2} - 28)\pi r^3.$$

I. *Solution by* E. B. Seitz.

Let x = the side of the hole, y = the distance of a plane parallel to the side of the hole from the centre of the sphere, u = the area of the section of the hole made by this plane, and V the required average.

Then $u = 2(r^2-y^2)\sin^{-1}\left\{\frac{x}{2(r^2-y^2)^{\frac{1}{2}}}\right\} + x(r^2-y^2-\frac{1}{4}x^2)^{\frac{1}{2}}$,

$$V = \frac{2\int_0^{r\sqrt{2}}\int_0^{\frac{1}{2}x} u\,dx\,dy}{\int_0^{r\sqrt{2}} dx} = \frac{\sqrt{2}}{r}\int_0^{r\sqrt{2}}\int_0^{\frac{1}{2}x} u\,dx\,dy$$

$$= \frac{\sqrt{2}}{6r}\int_0^{r\sqrt{2}}\left[(12r^2x-x^3)\sin^{-1}\left\{\frac{x}{(4r^2-x^2)^{\frac{1}{2}}}\right\}\right.$$
$$\left. + 16r^3\tan^{-1}\left(1-\frac{x^2}{2r^2}\right)^{\frac{1}{2}} - 4\pi r^3 + x^2(4r^2-2x^2)^{\frac{1}{2}}\right]dx$$

$$= \tfrac{1}{12}\pi r^3(23\sqrt{2}-28).$$

II. *Solution by the* Proposer.

Let r = radius of the sphere, $m = \frac{1}{2}r\sqrt{2}$, $2w$ = a side of the square hole, V = volume removed, and Δ = *average* volume removed. Then, by the solution of Question 4257, we have

$$V = \tfrac{4}{3}w^2(r^2-2w^2)^{\frac{1}{2}} + \tfrac{8}{3}w(3r^2-w^2)\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right)^{\frac{1}{2}} - \tfrac{8}{3}r^3\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right),$$

$$\Delta = \int_0^m V\,dw \div \int_0^m dw$$

$$= \frac{\sqrt{2}}{r}\left[\tfrac{1}{6}\sqrt{2}(6w^3-\tfrac{1}{2}r^2w)(\tfrac{1}{2}r^2-w^2)^{\frac{1}{2}} - \sqrt{2}\,r^4\sin^{-1}\left(\frac{w\sqrt{2}}{r}\right)\right.$$
$$\left. - \tfrac{8}{3}r^3 w\sin^{-1}\left(\frac{w^2}{r^2-w}\right) + \tfrac{8}{3}(6r^2w^2-w^4+3r^4)\sin^{-1}\left(\frac{w^2}{r^2-w^2}\right)^{\frac{1}{2}}\right]_0^{\frac{1}{2}r\sqrt{2}},$$

$$= \tfrac{1}{12}\pi r^3(23\sqrt{2}-28).$$

5829. (By Professor Cochez.) — Trouver les conditions pour que l'équation du quatrième degré $Ax^4+Cx^2+Dx+F=0$ ait trois racines égales. On trouve les 2 conditions

$$\left(\frac{3}{2}\right)^3\left(\frac{D}{A}\right)^2 = 12\frac{CF}{A^2} = -\frac{C^3}{A^3}.$$

I. *Solution by* J. A. Kealy, M.A.; J. Young, B.A.; *and others.*

Let a be the root repeated three times, than other root is $-3a$; hence

$$(x-a)^3(x+3a),\quad x^4+\frac{C}{A}x^2+\frac{D}{A}x+\frac{F}{A}$$ are identical expressions.

Equating coefficients, we have $-6a^2 = \frac{C}{A}$, $8a^3 = \frac{D}{A}$, $-3a^4 = \frac{F}{A}$;

therefore $$216a^6 = -\frac{C^3}{A^3} = 12\frac{CF}{A^2} = \frac{27}{8}\left(\frac{D}{A}\right)^2.$$

II. *Solution by the* Rev. J. L. Kitchin, M.A; R. Graham, M.A.; *and others.*

Consider the equation $ax^4+3bx^3y+6cx^2y^2+4dxy^3+ey^4=0$.

The conditions for its having three equal roots are

$$ae-4bd+3c^2=0,\quad ace+2bcd+ad^2-b^2e-c^3=0;$$

which can hardly be unknown, though I am unable to refer to any proof of them. An inspection of Prof. CAYLEY's expressions for these invariants in terms of the roots at once verifies them; or they may be obtained by transforming $ax^4+\ldots$ into $(a'x'+4b')x^3$, or thus:

Assume $$ax^4+\ldots=a(x+\alpha)(x+\beta)^3;$$

then $$4b=a(3\beta+\alpha),\quad 6c=a(3\beta^2+3\alpha\beta),$$

$$4d=a(\beta^3+3\alpha\beta^2),\quad e=a\alpha\beta^3;$$

whence $$4ae=4a^2\alpha\beta^3,\quad -16bd=-3a^2\alpha^2\beta^2-10a^2\alpha\beta^3-3a^2\beta^4,$$

and $$12c^2=3a^2\alpha^2\beta^2+6a^2\alpha\beta^3+3b^4,$$

giving $4(ae-4bd+3c^2)$ identically equal 0.

Again $$64\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}=\begin{vmatrix} 4 & \alpha+3\beta & 2(\alpha+\beta)\beta \\ \alpha+3\beta & 2(\alpha+\beta)\beta & (3\alpha+\beta)\beta^2 \\ 2(\alpha+\beta)\beta & (3\alpha+\beta)\beta^2 & 4\alpha\beta^3 \end{vmatrix}$$ in which, if the first column be multiplied, and the third divided, by β, the sum of the first and third columns is equal to twice the second; and the determinant consequently vanishes identically.

Putting $a=12A$, $b=0$, $c=2C$, $d=3D$, $e=12F$, the conditions, as found above, are $12AF+C^2=0$, $72ACF-27AD^2-2C^3=0$.

Subtracting the first, multiplied by 6C, from the second,

$$27AD^2+8C^3=0;$$

and this last, with the first, are the conditions as given in the Question.

5867. (By the Rev. W. A. WHITWORTH, M.A.) — Construct 40 different right-angled triangles having the common hypothenuse 32045, and all their sides integers.

I. *Solution by the* PROPOSER.

On the common hypothenuse = 32045 describe (Euclid I. 22) triangles having their other two sides equal to (i.) the first and last, (ii.) the 2nd and 79th, (iii.) the 3rd and 78th, and so on, of the 80 numbers registered in the subjoined table. The forty triangles thus described will be found to satisty the test of Euclid I. 48. They are therefore the triangles required.

716	7259	12325	18291	23067	27004	29848	31323
1363	7656	12920	19227	23205	27132	29920	31603
2277	7888	13572	19552	23693	27347	30381	31668
2400	8283	15080	19795	24124	27608	30420	31800
3045	8580	15708	20300	24205	27813	30821	31824
3757	8772	15916	21000	24795	27931	30875	31900
3955	10075	16269	21093	25200	28275	30956	31955
4901	10192	16704	21576	25389	29029	31059	31964
5304	11475	17051	22100	25636	29325	31117	32016
6764	11661	17253	22244	26312	29580	31212	32037

II. *Solution by the* Rev. G. H. HOPKINS, M.A.

Fermat has proved that every prime number of the form $4N+1$ is the sum of two square numbers; consequently every such prime number is the hypothenuse of a Pythagorean triangle.

Mr. Whitworth, in a paper on Pythagorean triangles (read before the Literary and Philosophical Society of Liverpool, Feb., 1875), has proved—The product of n prime hypothenuses, all different, will be itself the hypothenuse of 2^{n-1} Pythagorean triangles.

The prime factors of 32045 are 5, 13, 17, 29, all being of the form $4N+1$; hence the hypothenuses arising directly from 32045 will be 2^{4-1}.

Next, take each of the primes singly as factors of all the sides of the triangles, found by the product of the remaining three factors, viz., 5×6409, 13×2465, 17×1885, 29×1105; and we obtain $4 \times 2^{3-1}$ hypothenuses.

Thirdly, take the product of any two of the primes as factors of all the sides of the triangles found by the product of the remaining two factors, viz., 65×481, 85×377, 221×145, 481×65, 377×85, 145×221. By this arrangement, we obtain $\frac{4 \cdot 3}{1 \cdot 2} \times 2^{2-1}$ hypothenuses.

Lastly, take the product of any three of the primes as factors of all the sides of the triangles formed by the remaining factors, viz., 6409×5, 2465×13, 1885×17, 1105×29, this gives $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times 2^{1-1}$ hypothenuses.

In all, we obtain

$$2^{4-1} + 4 \cdot 2^{3-1} + \frac{4 \cdot 3}{1 \cdot 2} \cdot 2^{2-1} + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 2^{1-1},$$

or

$$\tfrac{1}{2}\left(2^4 + 4 \cdot 2^3 + \frac{4 \cdot 3}{1 \cdot 2} \cdot 2^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 2 + 1 - 1\right),$$

or

$$\tfrac{1}{2}(3^4 - 1), \text{ or } 40.$$

The general formula for a number with n prime factors of the form $4N+1$ is easily inferred from the above example; the number of hypothenuses which such a number would give is $\frac{1}{2}(3^n - 1)$.

To obtain the sides which correspond with the hypothenuse 32045, I must refer to the paper above mentioned, Art. 16, or to HEWLETT'S Fifth Edition of EULER'S *Algebra*, p. 390; and, to work out a single example, let

$$x^2 + y^2 = 32045 = 5 \cdot 13 \cdot 17 \cdot 29 = (2^2 + 1^2)(3^2 + 2^2)(4^2 + 1^2)(5^2 + 2^2),$$

$$(x + yi)(x - yi) = (2 + i)(2 - i)(3 + 2i)(3 - 2i)(4 + i)(4 - i)(5 + 2i)(5 - 2i).$$

Equating $x + iy$, where as usual, $i = \sqrt{-1}$, to four of the factors on the right-hand side of the identity, we can do it in 2^3 different ways. Take any one of these, thus

$$x + yi = (2 + i)(3 - 2i)(4 - i)(5 + 2i) = 179 + 2i,$$

or

$$x = 179 \text{ and } y = 2.$$

The sides given by these will be $179^2 - 2^2$ or 32037 and $2 \times 179 \times 2$ or 716. Similarly for other values.

The forty sets of sides which coincide with 32045 are

2277 : 31964.	17253 : 27004.	23067 : 22244.	31323 :	6764.
8283 : 30956.	21093 : 24124.	27813 : 15916.	32037 :	716.
1363 : 32016.	23693 : 21576.	27347 : 16704.	31117 :	7656.
7259 : 31212.	17051 : 27132.	27931 : 15708.	30821 :	8772.
11661 : 29848.	18291 : 26312.	25389 : 19552.	30381 :	10192.
3955 : 31800.	19795 : 25200.	24205 : 21000.	31955 :	2400.
10075 : 30420.	30875 : 8580.	16269 : 27608.	31059 :	7888.
11475 : 29920.	29325 : 12920.	4901 : 31668.	29029 :	13572.
3045 : 31900.	24795 : 20300.	3757 : 31824.	31603 :	5304.
19227 : 25636.	12325 : 29580.	28275 : 15080.	23205 :	22100.

These four groups are obtained from the four different arrangements of the factors of the given number as given above.

If it be required to find the number of Pythagorean triangles, not necessarily in their lowest terms, which have 32045 for one of the sides, we can proceed as follows :—

$x^2-y^2 = 5 \cdot 13 \cdot 17 \cdot 29$, this equation gives 2^{4-1} solutions.

$x^2-y^2 =$ any three of the factors, this would give in all $4 \cdot 2^{3-1}$.

$x^2-y^2 =$ any two of the factors, this would give in all $\frac{4 \cdot 3}{1 \cdot 2} \cdot 2^{2-1}$.

$x^2-y^2 =$ any one of the factors, this would give in all $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 2^{1-1}$.

The sum of these will be $\frac{1}{3}(3^4-1)$ or 40.

Also, if it be required to find the number of Pythagorean triangles having 2×32045 for one of the sides, which is always of the form $2ab$, we shall have again the number $\frac{1}{3}(3^4-1)$ or 40; a particular case of the general number $2a_1 a_2 a_3 \ldots a_n$, which I proved in the *Educational Times*, about five or six years back, to give $\frac{1}{3}(3^n-1)$ sides of the form $2ab$.

5911 (By the Rev. W. A. Whitworth, M.A.)—Show that the 80 acute angles of the 40 triangles of Question 5867 are obtained by rejecting all complete right angles from the values of the expression

$$180° + i\alpha + i'\beta + i''\gamma + i'''\delta;$$

where i, i', i'', i''' may each have any of the values -1, 0, $+1$; and where

$\alpha = 36° 52' \ldots$ $\beta = 22° 37' \ldots$ $\gamma = 28° 4' \ldots$ $\delta = 43\ 36' \ldots$

$= \tan^{-1}\frac{3}{4}$, $= \tan^{-1}\frac{5}{12}$, $= \tan^{-1}\frac{8}{15}$, $= \tan^{-1}\frac{33}{21}$.

[It may be remarked that, since each of the four i, i', i'', i''' is susceptible of three values, the expression will give 3^4 or 81 angles. One of these is, however, zero; the remaining 80 are the angles required.]

Solution by the Rev. G. H. Hopkins, M.A.

Let Z be such a number as 32045, which has all of its prime factors $z_1, z_2, z_3 \ldots z_n$ of the form $4N+1$.

Then, if $\theta_1 \theta_2 \theta_3 \ldots \theta_n$ be each of them one of the angles of the n Pythagorean triangles which have $z_1, z_2, z_3 \ldots z_n$ for the hypothenuses, the sides

will be $z_1 \cos\theta_1,\ z_2 \cos\theta_2,\ z_3 \cos\theta_3 \ldots z_n \cos\theta_n,$

$z_1 \sin\theta_1,\ z_2 \sin\theta_2,\ z_3 \sin\theta_3 \ldots z_n \sin\theta_n.$

Taking all the expressions similar to $z_r(\cos\theta_r \pm i\sin\theta_r)$, or, as it may be written, $z_r(\cos\epsilon_r\theta_r + i\sin\epsilon_r\theta_r)$, when i is $\sqrt{-1}$ and $\epsilon_r = \pm 1$; and forming the product of all these expressions, we obtain

$$Z\{\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n) + i\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots \epsilon_n\theta_n\},$$

which shows that, if Z be a given hypothenuse, then the sides are

$$Z\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n) \quad \text{and} \quad Z\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_n\theta_n).$$

Thus far we have taken $\epsilon_1, \epsilon_2, \epsilon_3 \ldots \epsilon_n$ each equal to ± 1; their values may be extended by putting them also equal to 0; for if $\epsilon_r = 0$, then z_r would be a common factor for all the sides of one triangle; hence $\epsilon_1, \epsilon_2, \epsilon_3 \ldots \epsilon_n$ can have any of the values -1, 0, $+1$; we must except the case where they are all equal to 0 for obvious geometrical reasons. We have, then, n quantities which can each of them have any one of 3 values; the number of combinations of these will be 3^n; and, allowing for the exception mentioned, we have $3^n - 1$, which gives $\frac{1}{2}(3^n - 1)$ triangles.

By hypothesis, all the n sides, of which $z_r\cos\theta_r$, $z_r\sin\theta_r$ are types, are integers; the two terms in the expression $z_r(\cos_r\theta_r + i\sin\epsilon_r\theta_r)$ are integers, it follows that in the product of these n expressions the two terms will be integers; therefore the sides of the $\frac{1}{2}(3^n - 1)$ triangles which have s for their hypotheneuse will be the $\frac{1}{2}(3^n - 1)$ integral values of

$$Z\cos(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots \epsilon_n\theta_n) \quad \text{and} \quad Z\sin(\epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots \epsilon_n\theta_n).$$

If the angle in the bracket is written in the more general form

$$k\,180^\circ + \epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots \epsilon_n\theta_n,$$

giving k such a value that the angle is not negative, and throwing out all right angles, we shall have the acute angles of the triangles. And if

$x_r^2 + y_r^2 = z_r^2$, then $\theta_r = \tan^{-1}\frac{x_r}{y_r}$.

Applying this to the particular number 32045, the prime factors are 5, 13, 17, 29; the sides corresponding to these four hypothenuses are 3, 4; 5, 12; 8, 15; 20, 21; and if $\alpha, \beta, \gamma, \delta$ be one of the acute angles in these four right-angled triangles

$$\alpha = \tan^{-1}\tfrac{3}{4},\ \beta = \tan^{-1}\tfrac{5}{12},\ \gamma = \tan^{-1}\tfrac{8}{15},\ \delta = \tan^{-1}\tfrac{20}{21}.$$

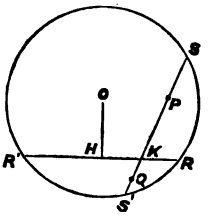
Mr. WHITWORTH gives the values of these angles, and as their sum lies between 90° and 180°, in the general expression we may put $k = 1$, or all the acute angles of the $\frac{1}{2}(3^4 - 1)$ triangles are contained in the formula $180^\circ + \epsilon_1\alpha + \epsilon_2\beta + \epsilon_3\gamma + \epsilon_4\delta$.

For various obscure points in this Solution, I must refer to my solution of Quest. 5867, and Mr. WHITWORTH's pamphlet there mentioned.

5872 & 5891. (By E. B. SEITZ and S. ROBERTS, M.A.)—A line is drawn at random across a circle, and a point is taken at random in each segment; show that (1) the average distance between the points is $\frac{45}{128}\pi r$, and (2) the average length of the chord is $\frac{75}{128}\pi r$.

I. *Solution by* SAMUEL ROBERTS, M.A.

1. Let RR'SS' be a circle with centre O and radius r, R'R a random line dividing the circle into the segments R'RS, R'RS', represented by A, B respectively. Take P a random point in A, and Q a random point in B, and let the chord through them meet the circle in S, S', and the random line in K. Draw OH = k to the middle point H of RR'; and put HR = l, HK = x, KP = y, KS = Y, KQ = z, KS' = Z, $\angle$ RKP = θ. We need only consider random lines parallel to R'R, for these become any other parallel system by turning the whole figure through an angle.



The element of the circular surface at P is $dx \,.\, dy \,.\, \sin\theta$, that at Q referred to the point P is $(y+z)\, dz\, d\theta$, and the required average is therefore

$$\int_{-r}^{+r}\int_{-l}^{+l}\int_{0}^{\pi}\int_{0}^{Y}\int_{0}^{Z}(y+z)^2\, dk \,.\, dx \,.\, \sin\theta\, d\theta \,.\, dy \,.\, dz \div \int_{-r}^{+r} A \,.\, B \,.\, dk \quad \ldots\ldots(a),$$

The integral in the denominator is most simply obtained by the chance that P, Q are on opposite sides of the random line (Williamson's *Int. Calc.*, p. 340, ex. 40), viz.,

$$2\int_{-r}^{+r} A \,.\, B \,.\, dk \div 2r \,.\, (\pi r^2)^2 = \frac{128}{45\pi^2}; \quad \therefore \int_{-r}^{+r} A \,.\, B \,.\, dk = \frac{128}{45} r^5.$$

For (a) we have,

$$\frac{15}{256 r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_{0}^{\pi} YZ\left\{2\,(Y+Z)^2 - YZ\right\} dk \,.\, dx \,.\, \sin\theta \,.\, d\theta.$$

But $$Y = \left\{r^2 - (x\sin\theta + k\cos\theta)^2\right\}^{\frac{1}{2}} + k\sin\theta - x\cos\theta,$$

and $$Z = \left\{r^2 - (x\sin\theta + k\cos\theta)^2\right\}^{\frac{1}{2}} - k\sin\theta + x\cos\theta;$$

$$\therefore \frac{15}{256 r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_{0}^{\pi}\left\{7\,(l^2-x^2)^2 + 8\,(x\cos\theta - k\sin\theta)^2\,(l^2-x^2)\right\} dk\, dx \sin\theta\, d\theta$$

$$= \frac{15}{128 r^5}\int_{-r}^{+r}\int_{-l}^{+l}\left\{7\,(l^2-x^2)^2 + \tfrac{8}{3}\,(l^2-x^2)\,x^2 + \tfrac{16}{3}\,(l^2-x^2)\,k^2\right\} dk\, dx$$

$$= \frac{15}{64 r^5}\int_{-r}^{+r}\left(\frac{184 l^5 + 160 l^3 k^2}{45}\right) dk = \frac{45}{128}\pi r.$$

2. Here the required average is

$$\frac{\displaystyle\int_{-r}^{+r}\int_{-l}^{+l}\int_{0}^{\pi}\int_{0}^{Y}\int_{0}^{Z}(Y+Z)\,(y+z)\, dk\, dx\, \sin\theta\, d\theta\, dy\, dz}{\displaystyle\int_{-r}^{r}\left(\pi r^2 + kl - r^2\cos^{-1}\frac{k}{r}\right)\left(r^2\cos^{-1}\frac{k}{r} - kl\right) dk}$$

$$= \tfrac{1}{2}\int_{-r}^{+r}\int_{-l}^{+l}\int_{0}^{\pi} YZ\,(Y+Z)^2\, dk\, dx\, \sin\theta\, d\theta + (\tfrac{1}{2}\tfrac{3}{5}\tfrac{2}{5} r^5).$$

But $Y = \{r^2-(x\sin\theta+k\cos\theta)^2\}^{\frac{1}{2}}+k\sin\theta-x\cos\theta,$

and $Z = \{r^2-(x\sin\theta+k\cos\theta)^2\}^{\frac{1}{2}}-k\sin\theta+x\cos\theta;$

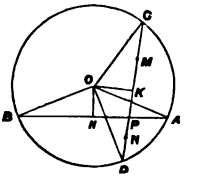
hence the required average is

$$\frac{45}{64r^5}\int_{-r}^{+r}\int_{-l}^{+l}\int_0^{\pi}(l^2-x^2)\{r^2-(x\sin\theta+k\cos\theta)^2\}\,dk\,dx\,\sin\theta\,d\theta$$

$$=\frac{45}{32r^5}\int_{-r}^{+r}\int_{-l}^{+l}\left\{(l^2-x^2)\,r^2-\tfrac{2}{3}(l^2-x^2)\,x^2-\tfrac{1}{3}(l^2-x^2)\,k^2\right\}dk\,dx$$

$$=\frac{1}{16r^5}\int_{-r}^{+r}\left\{\frac{13l^3+10l^2k^2}{8}\right\}dk=\tfrac{7}{128}\pi r.$$

II. *Solution by* E. B. Seitz.



1. Let AB be the chord formed by the random line, M, N the random points, CD the chord through them, and O the centre of the circle. Draw OH and OK perpendicular to AB and CD. Let $OA=r$, $PM=x$, $MN=y$, $PC=u$, $PD=v$, $\angle AOH=\theta$, $\angle HOK=\phi$, and $\angle COK=\psi$. Then we have

$$CD = 2r\sin\psi,$$

$$\text{area seg. ADB} = r^2(\theta-\sin\theta\cos\theta),$$

$$\text{area seg. ACB} = r^2(\pi-\theta+\sin\theta\cos\theta),$$

$$u = r(\sin\psi+\cos\theta\operatorname{cosec}\phi-\cot\phi\cos\psi),$$

and

$$v = r(\sin\psi-\cos\theta\operatorname{cosec}\phi+\cot\phi\cos\psi).$$

An element of the circle at M is $r\sin\psi\,d\psi\,dx$, at N it is $y\,d\phi\,dy$, and at H an element of the distance across the circle in the direction HO is $r\sin\theta\,d\theta$. The limits of θ are 0 and $\frac{1}{2}\pi$; of ϕ, 0 and $\frac{1}{2}\pi$, and doubled; of ψ, $\theta-\phi$ and $\theta+\phi$ when $\phi<\theta$, and $\phi-\theta$ and $\phi+\theta$ when $\phi>\theta$. But in the integration with respect to ψ we will have powers of $\cos\psi$ only; hence, since

$$\cos(\theta-\phi) = \cos(\phi-\theta),$$

we may take $\theta-\phi$ and $\theta+\phi$ as the limits of ψ for all values of ϕ. The limits of x are 0 and u; and of y, x and $v+x$.

Therefore the average distance between the points is

$$\Delta = \frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x} r\sin\theta\,d\theta\,d\phi\,r\sin\psi\,d\psi\,dx\,y^2\,dy}{\int_0^{\frac{1}{2}\pi} r^5(\theta-\sin\theta\cos\theta)(\pi-\theta+\sin\theta\cos\theta)\sin\theta\,d\theta}$$

$$=\frac{45}{32r^5}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x}\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi\,dx\,y^2\,dy$$

$$=\frac{15}{32r^5}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}[(v+x)^3-x^3]\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi\,dx$$

$$= \frac{15}{128r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\{(u+v)^4-(u^4+v^4)\}\sin\theta\sin\psi\,d\theta\,d\phi\,d\psi$$

$$= \frac{r}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(10-3\sin^2\theta-10\cos^2\phi+12\sin^2\theta\cos^2\phi)\sin^4\theta\sin\phi\,d\theta\,d\phi$$

$$= \frac{r}{12}\int_0^{\frac{1}{2}\pi}(20+3\sin^2\theta)\sin^4\theta\,d\theta = \tfrac{45}{128}\pi r.$$

2. The average length of the chord through the points is

$$L = \frac{2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x}2r\sin\psi\,r\sin\theta\,d\theta\,d\phi\,r\sin\psi\,d\psi\,dx\,y\,dy}{\int_0^{\pi}r^5(\theta-\sin\theta\cos\theta)(\pi-\theta+\sin\theta\cos\theta)\sin\theta\,d\theta}$$

$$= \frac{45}{16r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\int_x^{v+x}\sin\theta\sin^2\psi\,d\theta\,d\phi\,d\psi\,dx\,y\,dy$$

$$= \frac{45}{32r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}\int_0^{u}\{(v+x)^2-x^2\}\sin\theta\sin^2\psi\,d\theta\,d\phi\,d\psi\,dx$$

$$= \frac{45}{16r}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\theta-\phi}^{\theta+\phi}uv\sin\theta\sin^3\psi\,d\theta\,d\phi\,d\psi$$

$$= \frac{3r}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(5-\sin^2\theta-5\cos^2\phi+6\sin^2\theta\cos^2\phi)\sin^4\theta\sin\phi\,d\theta\,d\phi$$

$$= \frac{r}{4}\int_0^{\frac{1}{2}\pi}(10+3\sin^2\theta)\sin^4\theta\,d\theta = \tfrac{75}{128}\pi r.$$

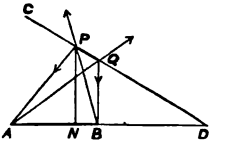
5904. (By Professor MINCHIN, M.A.)—Iron filings are scattered over a sheet of paper on which rests a magnet. Prove that the locus of those filings which dip towards the same point in the line of the magnet is a circle. [Magnetism is supposed to be concentrated at the poles.]

I. *Solution by* H. L. ORCHARD, B.A., L.C.P.; Rev. J. L. KITCHIN, M.A.; E. RUTTER; *and others.*

If P, P′ are the poles of, and Q the fixed point on, the magnet, while O is one of the filings dipping towards Q, the condition $\frac{\sin POQ}{OP^2} = \frac{\sin P'OQ}{OP'^2}$

must be fulfilled; but, obviously, sin POQ : sin P'OQ = $\frac{PQ}{OP} : \frac{P'Q}{OP'}$ so that OP : OP' = $(PQ)^{\frac{1}{3}} : (P'Q)^{\frac{1}{3}}$, and the locus of O is the circle whose diameter is the segment between points in which PP' is divided in the subtriplicate ratio of PQ to P'Q.

II. *Solution by* Rev. G. H. Hopkins, M.A.; Prof. Evans, M.A.; *and others.*



This theorem resolves itself into the following:—A and B are two centres of force which act upon P and Q, the extremities of a small rigid bar, in such a way that A attracts B and repels Q, and B attracts Q and repels P; the law of force being some power of the distance (d^n); then, if PQ is at rest, with its direction cutting AB produced in a fixed point D, when PQ becomes very small, the locus of its positions of equilibrium will be a circle.

The small rigid bar PQ is in equilibrium under the action of the forces from A and B; the sum of the moments of the forces acting on P round Q is

$$\phi(AB)\sin APC - \phi(BP)\sin BPD = 0,$$

where $\phi(d)$ is put for the law of force. Therefore

$$\frac{\phi(AP)}{\phi(BP)} = \frac{\sin(PBA-PDA)}{\sin(PAB+PDA)} = \frac{\sin PBA - \cos PBA\tan PDA}{\sin PAB + \cos PAB\tan PDA}$$

$$= \frac{\frac{PN}{PB} - \frac{BN}{PB}\cdot\frac{PN}{DN}}{\frac{PN}{PA} + \frac{PN}{PA}\cdot\frac{PN}{DN}};$$

therefore $$\frac{PB\phi(AP)}{PA\phi(BP)} = \frac{1-\frac{BN}{DN}}{1+\frac{AN}{DN}} = \frac{DB}{DA} = \text{a constant ratio.}$$

Assume that the law of force is as the n^{th} power of the distance, then

$$\frac{PB\phi(AP)}{PA\phi(BP)} \propto \left(\frac{AP}{BP}\right)^{n-1}, \quad \text{therefore} \frac{AP}{BP} = \text{a constant } (k).$$

If the middle point of AB be the origin, and AB lie in the axis of x,

$$\frac{y^2+(x+a)^2}{y^2+(x-a)^2} = k^2, \quad \text{or} \quad y^2+x^2+2\frac{1+k^2}{1-k^2}ax+a^2 = 0 \text{ is the required locus.}$$

This locus is independent of the value of n, and is therefore true for magnetism where $n = -2$.

5627. (By Professor Townsend, F.R.S.)—A cylinder of revolution of uniform density being supposed to float, with axis vertical, in a gravitating fluid of greater density; show, by any method, that the coaxal spheroid

of revolution, the square of whose equatoreal is double of that of its polar radius, will intersect its surface in the limiting circle of floatation consistent with the stability of its equilibrium under the action of gravity.

Solution by the PROPOSER.

Denoting by h and r the height and radius of the cylinder, and by z the depth of its immersion in the fluid; then, since, on well-known principles, the elevation of the metacentre corresponding to the position of floatation above the centre of inertia of the cylinder $= \frac{r^2}{4z} - \frac{h-z}{2}$, therefore, for the limiting value of z consistent with the stability of its equilibrium under the action of gravity, $2z(h-z) = r^2$; and therefore, &c.

5772. (By C. LEUDESDORF, M.A.)—If the areas of the pedals of any given closed curve with regard to four points A, B, C, P in its plane, taken successively as pedal origin, be denoted by (A), (B), (C), (P) respectively, prove that $(P) = x(A) + y(B) + z(C) + \frac{1}{2}\pi t^2$,
where x, y, z are the areal coordinates of P referred to the triangle ABC, and t is the length of the tangent from P to the circle around ABC.

Solution by W. J. C. SHARP, B.A.; J. A. KEALY, M.A.; *and others.*

If S be the area of the minimum pedal, the origin of which is chosen as origin of rectangular Cartesian coordinates, and A, B, C, P denote the points (ab), (a_1b_1), (a_2b_2), P (xy), we have (Williamson's *Int. Calc.*, p. 183)

$$(A)-S = \tfrac{1}{2}\pi(a^2+b^2), \quad (B)-S = \tfrac{1}{2}\pi(a_1^2+b_1^2),$$

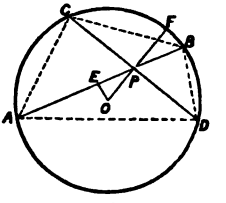
$$(C)-S = \tfrac{1}{2}\pi(a_2^2+b_2^2), \quad (P)-S = \tfrac{1}{2}\pi(\xi^2+\eta^2);$$

$$\begin{vmatrix} \xi^2+\eta^2 & \xi & \eta & 1 \\ a^2+b^2 & a & b & 1 \\ a_1^2+b_1^2 & a_1 & b_1 & 1 \\ a_2^2+b_2^2 & a_2 & b_2 & 1 \end{vmatrix} = \Delta t^2, \quad \text{therefore} \quad \begin{vmatrix} (P) & \xi & \eta & 1 \\ (A) & a & b & 1 \\ (B) & a_1 & b_1 & 1 \\ (C) & a_2 & b_2 & 1 \end{vmatrix} = \tfrac{1}{2}\pi\Delta t^2;$$

therefore $(P)\,\Delta = (A)\,\Delta(BPC) + (B)\,\Delta(APC) + (C)\,\Delta(APB) + \frac{1}{2}\pi\Delta\mu^3$,
where Δ stands for the triangle ABC, therefore, &c.

5168. (By ARTEMAS MARTIN, M.A.)—Through a point taken at random in the surface of a circle, two chords are drawn, one at random, and the other at right angles to the radius through that point; show that the average area of the quadrilateral formed by joining the ends of the chords is one-fourth of the area of the circle.

Solution by the PROPOSER; Professor NASH, M.A.; *and others.*



Let O be the centre of the circle, P the random point in its surface, AB the random chord, and CD the chord at right angles to the radius through P. Draw OE perpendicular to AB. Let $OP = x$ and $\angle APO = \phi$; then

$$\angle APC = \tfrac{1}{2}\pi - \phi, \quad CD = 2(r^2 - x^2)^{\frac{1}{2}},$$
$$AB = 2(r^2 - x^2\sin^2\phi)^{\frac{1}{2}};$$

and the area of the quadrilateral

$$ACBD = AB \times CD \times \tfrac{1}{2}\sin APC = 2(r^2 - x^2)^{\frac{1}{2}}(r^2 - x^2\sin^2\phi)^{\frac{1}{2}}\cos\phi.$$

If Δ be the average area required, we have

$$\Delta = \frac{2\int_0^r\int_0^{\frac{1}{2}\pi}(r^2-x^2)^{\frac{1}{2}}(r^2-x^2\sin^2\phi)^{\frac{1}{2}}\cos\phi \times 2\pi x\,dx\,d\phi}{\int_0^r\int_0^{\frac{1}{2}\pi}2\pi x\,dx\,d\phi}$$

$$= \frac{8}{\pi r^2}\int_0^r\int_0^{\frac{1}{2}\pi}(r^2-x^2)^{\frac{1}{2}}(r^2-x^2\sin^2\phi)^{\frac{1}{2}}\,x\,dx\cos\phi\,d\phi$$

$$= \frac{4}{\pi r^2}\int_0^r(r^2-x^2)\,x\,dx + \frac{4}{\pi}\int_0^r(r^2-x^2)^{\frac{1}{2}}\sin^{-1}\left(\frac{x}{r}\right)dx$$

$$= \frac{r^2}{\pi} + \frac{1}{\pi}\left[2x(r^2-x^2)^{\frac{1}{2}}\sin^{-1}\left(\frac{x}{r}\right) + r^2\left(\sin^{-1}\frac{x}{r}\right)^2 - x^2\right]_0^r$$

$$= \tfrac{1}{4}\pi r^2.$$

5728. (By F. P. MATZ, M.A.)—An equilateral spherical triangle is (1) inscribed in, (2) circumscribed about, a circle of radius ϕ; show that the cosine of an angle of the triangle is $\dfrac{3\cos^2\phi - 1}{3\cos^2\phi + 1}$ in (1), and $\frac{3}{2}\cos^2\phi - 1$ in (2).

Solution by Rev. J. L. KITCHIN, M.A.; R. KNOWLES, L.C.P.; *and others.*

1. Let ABC be the spherical triangle; P the pole of the circle; and PD a perpendicular to AB. Denote the arc AP by ϕ, and let $\angle PAD = PAC = \frac{1}{2}\beta$. But the $\angle APD = \frac{1}{3}\pi$; consequently

$$\cot\tfrac{1}{2}\beta = \cos\phi\tan\tfrac{1}{3}\pi = \cos\phi\sqrt{3}, \quad \therefore \cos\beta = \frac{\cot^2\frac{1}{2}\beta - 1}{\cot^2\frac{1}{2}\beta + 1} = \frac{3\cos^2\phi - 1}{3\cos^2\phi + 1}.$$

2. Let $PD = \phi$; then, by well known formulæ,

$$\cos\tfrac{1}{2}\beta = \cos\phi\sin\tfrac{1}{3}\pi = \tfrac{1}{2}\cos\phi\sqrt{3}, \quad \text{therefore } \cos\beta = \tfrac{3}{2}\cos^2\phi - 1.$$

5390. (By Professor LLOYD TANNER, M.A.)—When an equation $F(s, p, q, z, x, y) = 0$ has two independent first integrals, each involving an arbitrary function, it can be reduced to the form $\frac{d^2\zeta}{dx\,dy} = 0$, where ζ is a function of z, x, y.

Solution by the PROPOSER.

It is clear that no first integral of

$$F(s, p, q, z, x, y) = 0 \quad \ldots\ldots(1)$$

can involve both p, q. For the x-derivative of such an expression would involve r, s; the y-derivative would involve s, t; and it would be impossible from these to obtain (1), which involves neither r nor t. Say, then, a first integral involves p but not q. We can form (1) by differentiating with respect to y, and in the process we can eliminate an arbitrary function of x. Hence such a first integral may be put into the form $f(p, z, x, y) = \phi(x)$, and (1) is equivalent to $\frac{df}{dp}s + \frac{df}{dz}q + \frac{df}{dy} = 0$ ...(2).

If there be another first integral, independent of the one above, it must be of the form $f_1(q, z, x, y) = \psi(y)$;

so that (1) may be written $\frac{df_1}{q}s + \frac{df_1}{z}p + \frac{df_1}{dx} = 0 \quad \ldots\ldots(3)$.

Suppose, now, (1) is solved for s; then, by (2), it will be linear in q, since f does not involve q. Also, by (3), it will be linear in p, since f_1 does not involve p. Hence, (1) may be written

$$s + Apq + Pp + Qq + Z = 0 \quad \ldots\ldots(4),$$

A, P, Q, Z being functions of x, y, z.

On comparing (4) with (2), we get

$$(Ap + Q)\frac{df}{dp} = \frac{df}{dz}, \quad (Pp + Z)\frac{df}{dp} = \frac{df}{dy};$$

similarly $$(AP + P)\frac{df_1}{dq} = \frac{df_1}{dz}, \quad (Qq + Z)\frac{df_1}{dq} = \frac{df_1}{dx}.$$

Eliminating f, f_1 from these equations, we find

$$\frac{dA}{dy} = \frac{dP}{dz}, \quad \frac{dA}{dx} = \frac{dQ}{dz}, \quad \frac{dP}{dx} = \frac{dQ}{dy} = AZ + \frac{dZ}{dz} - PQ \quad \ldots\ldots(5),$$

and these are, as we proceed to show, sufficient to ensure the possibility of the indicated transformation.

In virtue of the first three conditions, $A\,dz + P\,dy + Q\,dx$ is an exact differential, $= d\mathrm{M}$ say; and

$$\frac{d\mathrm{M}}{dz} = A, \quad \frac{d\mathrm{M}}{dy} = P, \quad \frac{d\mathrm{M}}{dx} = Q \quad \ldots\ldots(6).$$

Also we can find a function $f(x, y, z)$ such that

$$\frac{df}{dz} = e^{\mathrm{M}}, \quad \frac{d^2f}{dx\,dy} = Ze^{\mathrm{M}} \quad \ldots\ldots(7),$$

for the only condition that this may be possible is

$$\frac{d}{dz}Ze^{M} = \frac{d^2e^{M}}{dx\,dy}, \text{ or } \frac{dZ}{dz} + AZ = \frac{dP}{dx} + PQ,$$

which is included in (5). This function f we take as our new dependent variable ζ. We have

$$\begin{aligned}\frac{d^2\zeta}{dx\,dy} &= \frac{d^2f}{dx\,dy} + p\frac{d^2f}{dy\,dz} + q\frac{d^2f}{dx\,dz} + pq\frac{d^2f}{dz^2} + s\frac{df}{dz}\\ &= \{Z + Pp + Qq + Apq + s\}e^{M}, \text{ by (6), (7)}\\ &= 0, \text{ by (4)}.\end{aligned}$$

It may be noticed that (5) form a generalization of Euler's conditions of integrability of (4) when A vanishes.

5033. (By S. Tebay, B.A.)—The roots of a cubic equation are x_1, x_2, x_3. Given $x_1 + x_2 + x_3 = a$, and $x_1 > x_2 > x_3$; show that the mean form of the equation is $x^3 - ax^2 + \frac{1}{4}a^2 - \frac{1}{60}a^3 = 0$.

Solution by the Proposer; F. C. Mathews, M.A.; *and others.*

We have to calculate the mean values of $x_2x_3 + x_3x_1 + x_1x_2$ and $x_1x_2x_3$. Let $t = x_1 + x_2$; then the first of these is $t(x_2 + x_3) - x_2^2$. Multiply by dx_2, and integrate from $x_2 = x_3$ to $x_2 = \frac{1}{2}t$, and then put $a - x_3$ for t, and we find

$$\tfrac{1}{12}a^3 + \tfrac{1}{4}a^2x_3 - \tfrac{2}{4}ax_3^2 + \tfrac{2}{4}x_3^3.$$

Multiply this by dx_3, and integrate from $x_3 = 0$ to $x_3 = \frac{1}{3}a$, and we get $\frac{1}{48}a^4$. But the number of solutions is $\frac{1}{12}a^2$ (see Quest. 4616). Therefore the average is $\frac{1}{4}a^2$. Again, the mean value of $x_1x_2x_3$ is

$$\frac{12}{a^2}\int_0^{\frac{1}{3}a} x_3\,dx_3 \int_{x_3}^{\frac{1}{2}t} x_2\,dx_2\,(t - x_2) = \frac{1}{a^2}\int_0^{\frac{1}{3}a} x_3\,dx_3\,(a^3 - 3a^2x_3 - 3ax_3^2 + 9x_3^3) = \frac{a^3}{60}.$$

Hence the mean form of the cubic is as given in the Question.

5923. (By D. Edwardes.)—If $x + y + z = 0$, prove that, if x, y, z be any real quantities, the expression $x^3y^3 + y^3z^3 + z^3x^3$ cannot be positive.

Solution by T. Mitcheson, B.A., L.C.P.; J. O'Regan; *and others.*

Since $x^3(y^3 + z^3) = x^3(y + z)(y^2 + z^2 - yz) = -x^4(y^2 + z^2 - yz)$ is negative, because $y^2 + z^2 > yz$ if y, z real; hence $x^3y^3 + y^3z^3 + z^3x^3$ = a negative.

5612. (By S. Roberts, M.A.)—Prove that (1) the locus of points such that the sum of the squares of their normal distances from a given conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ is constant, is a coaxial and concentric conicoid; (2) if the constant sum is $2(a^2+b^2+c^2)$, the locus becomes a cone; and (3) for a particular value of the constant sum it is identical with the given conicoid, if this is one of a certain family of real similar ellipsoids.

Solution by the Proposer.

To obtain the equation of the locus, we have to find the coefficient of k^{10} in the equation of the parallel where k is the modulus; but since we know that the equation contains x^2, y^2, z^2 only, and linearly, we may suppose $z=0$ in the first instance, and complete by symmetry. Putting $z=0$, we have

$$\left\{(a^2-b^2)^2k^8-2(a^2-b^2)\left[a^4-b^4+(a^2-2b^2)x^2+(2a^2-b^2)y^2\right]k^6+\&c.\right\}$$
$$\times\left\{\frac{c^2}{c^2-a^2}x^2+\frac{c^2}{c^2-b^2}y^2+c^2-k^2\right\}^2=0.$$

And the coefficient of k^{10} is

$$-2(a^2-b^2)\left\{\frac{2a^2c^2+2a^2b^2-3b^2c^2-a^4}{c^2-a^2}x^2+\frac{3a^2c^2-2b^2c^2-2a^2b^2+b^4}{c^2-b^2}y^2\right.$$
$$\left.+(a^2-b^2)(a^2+b^2+c^2)\right\}.$$

Therefore, for the equation of the parallel, we have

$$(a^2-b^2)^2(b^2-c^2)^2(c^2-a^2)^2k^{12}$$
$$-2k^{10}(a^2-b^2)(b^2-c^2)(c^2-a^2)\left\{\begin{array}{l}(2a^2c^2+2a^2b^2-3b^2c^2-a^4)(b^2-c^2)x^2\\+(2b^2c^2+2b^2a^2-3a^2c^2-b^4)(c^2-a^2)y^2\\+(2a^2c^2+2b^2c^2-3a^2b^2-c^4)(a^2-b^2)z^2\\+(a^2-b^2)(b^2-c^2)(c^2-a^2)(a^2+b^2+c^2)\end{array}\right\}.$$
$$+Pk^8+\&c.=0.$$

Writing A, B, C for a^2, b^2, c^2 respectively, the equation of the locus is

$$\frac{2}{(A-B)(B-C)(C-A)}\left\{\begin{array}{l}(2AC+2AB-3BC-A^2)(B-C)x^2\\+(2BC+2AB-3AC-B^2)(C-A)y^2\\+(2AC+2BC-3AB-C^2)(A-B)z^2\end{array}\right\}$$
$$+2(A+B+C)=K^2,$$

where K^2 is the constant sum. If $K^2=2(A+B+C)$ this is a cone.

If the surface is similar to the original conicoid, we have

$$(2AC+2AB-3BC-A^2)(B-C)A=(2BC+2AB-3AC-B^2)(C-A)B$$
$$=(2AC+2BC-3AB-C^2)(A-B)C,$$

from which we derive

$$\left.\begin{array}{r}A^2(B+C)+B^2(A+C)+C^2(A+B)-15ABC=0\\2(A^2+B^2+C^2)-5(AB+BC+CA)=0\end{array}\right\}\ \ldots\ldots\ldots(a).$$

And if $A_1B_1C_1$ is a common solution, the complete set is obtained by permuting these values.

If we eliminate C from (*a*), we get an equation of the 6th degree as to

A : B; but, writing p for $\frac{(A+B)^2}{AB}$, this reduces to a cubic; viz.,

$$p^3-21p^2+138p-289=0;$$

or, writing $q+7$ for p, $\quad q^3-9q-9=0,$

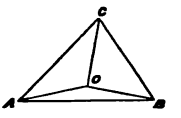
an equation whose roots are real, lying between -3 and -2, -2 and -1, and $+3$ and $+4$. Hence the roots p lie between 4 and 5, 5 and 6, 10 and 11, and the quadratic $\quad (A+B)^2-ABp=0$

has real positive roots. Hence the ratios are real and positive, and K^2 can be determined so that the locus shall coincide with the original conicoid. A similar result holds in plane space.

5809. (By the Editor.)—If O be any point within a triangle ABC, prove that $OA^n \sin 2A + OB^n \sin 2B + OC^n \sin 2C$ is least when O is the centre of the circumscribed circle.

Solution by E. B. Seitz.

Let $OA=x$, $OB=y$, $OC=z$, $BC=a$, $CA=b$, $AB=c$, $\angle BOC=\theta$, $\angle COA=\phi$, $\angle AOB=\psi$, and $\angle OCB=\omega$. Then we have



$$x^n \sin 2A + y^n \sin 2B + z^n \sin 2C = \text{a minimum} \quad \ldots(1),$$

$$x^2 = b^2 + z^2 - 2bz\cos(C-\omega) \ldots\ldots\ldots\ldots\ldots(2),$$

$$y^2 = a^2 + z^2 - 2az\cos\omega \ldots\ldots\ldots\ldots\ldots(3),$$

$$a\sin\omega = y\sin\theta, \quad b\sin(C-\omega) = x\sin\phi \ldots\ldots\ldots\ldots\ldots(4, 5).$$

Differentiating (1), (2), (3), regarding z as constant, we have

$$x^{n-1}\sin 2A\,dx + y^{n-1}\sin 2B\,dy = 0 \ldots\ldots\ldots\ldots\ldots(6),$$

$$x\,dx = -bz\sin(C-\omega)\,d\omega, \quad y\,dy = az\sin\omega\,d\omega \ldots\ldots\ldots\ldots\ldots(7, 8).$$

Substituting these values of dx and dy in (6), we find

$$\frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\phi}{y^{n-1}\sin 2B}.$$

Regarding y as constant, we find, by a similar process,

$$\frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\psi}{z^{n-1}\sin 2C};$$

therefore $\quad \frac{\sin\theta}{x^{n-1}\sin 2A} = \frac{\sin\phi}{y^{n-1}\sin 2B} = \frac{\sin\psi}{z^{n-1}\sin 2C} \quad \ldots\ldots\ldots\ldots\ldots(9).$

Now equations (9) are satisfied by the conditions $x=y=z$, and $\theta=2A$, $\phi=2B$, $\psi=2C$; hence, for a minimum, O must be the centre of the circumscribed circle.

5859. (By J. J. Walker, M.A.)—If α, β, γ are any three vectors, show that $(T\alpha)^2 S\beta\gamma + S\gamma\alpha S\alpha\beta = S(V\alpha\beta \cdot V\alpha\gamma)$, and verify the result trigonometrically.

Solution by E. Anthony, M.A.; Edward Rutter; *and others.*

We have $V\alpha\beta = -V\beta\alpha = -\beta\alpha + S\alpha\beta$, $V\alpha\gamma = \alpha\gamma - S\alpha\gamma$;

hence $V\alpha\beta \cdot V\alpha\gamma = -\beta\alpha^2\gamma + \beta_1 S\alpha\gamma + \alpha\gamma S\alpha\beta - S\alpha\beta \cdot S\alpha\gamma$,

and $S(V\alpha\beta \cdot V\alpha\gamma) = -\alpha^2 S\beta\gamma + S\beta\alpha \cdot S\alpha\gamma + S\alpha\gamma \cdot S\alpha\beta - S\alpha\beta \cdot S\alpha\gamma$

$= -\alpha^2 S\beta\gamma + S\alpha\beta \cdot S\alpha\gamma$.

If OA, OB, OC are the vectors α, β, γ respectively, and a, b, c the arcs joining the points in which they meet a sphere having O as centre, then we have $(T\alpha)^2 S\beta\gamma = -OA^2 \cdot OB \cdot OC \cos a$,

and $S\alpha\beta \cdot S\gamma\alpha = OA^2 \cdot OB \cdot OC \cos b \cos c$,

so that $(T\alpha)^2 S\beta\gamma + S\gamma\alpha \cdot S\alpha\beta = -OA^2 \cdot OB \cdot OC (\cos a - \cos b \cos c)$

$= -OA^2 \cdot OB \cdot OC \sin b \sin c \cos A$.

Now $V\alpha\beta$, $V\alpha\gamma$ are vectors perpendicular to the planes of OA, OB and OA, OC respectively, of lengths $OA \cdot OC \sin b$, $OA \cdot OB \sin c$ respectively. and containing an angle equal to A.

Hence, also, $S(V\alpha\beta \cdot V\alpha\gamma) = -OA^2 \cdot OB \cdot OC \sin b \sin c \cos A$.

5956. (By Professor Townsend, F.R.S.)—A slender uniform circular ring being supposed to attract, according to the law of the inverse cube of the distance, a material particle situated anywhere in its space; show that the attraction:—

(*a*) For all positions of the particle, passes in direction through the vertex of the cone which envelopes along the circumference of the ring its sphere of connection with the position of the particle.

(*b*) For all positions of the particle, varies in magnitude directly as the distance from the centre, and inversely as the square of the product of the greatest and least distances from the circumference of the ring.

(*c*) For positions of the particle limited to a definite sphere passing through the circumference of the ring, varies in magnitude directly as the distance from the fixed centre through which it then passes, and inversely as the square of the distance from the plane of the ring.

Solution by the Proposer.

Of these properties the first and second, from which the third is an obvious corollary, may be easily proved geometrically, but inferred perhaps more readily from the easily obtained value of the potential V of the mass, for any point in its space, corresponding to the law of force, as follows:—

Denoting by a and by m the radius and the mass of the ring, by x and y the coordinates of the position P of the particle perpendicular respec-

tively to the axis and to the plane of the ring, by r the distance of P from the centre, and by r_1 and r_2 its greatest and least distances from the circumference of the ring; then since, as may be proved at once without difficulty, $V = -\frac{1}{2}m[(x+a)^2+y^2]^{-\frac{1}{2}}[(x-a)^2+y^2]^{-\frac{1}{2}} = -\frac{1}{2}\frac{m}{r_1 r_2}$,

therefore at once, by differentiation with respect to x and to y,

$$\frac{dV}{dx} = m \cdot \frac{x^2+y^2-a^2}{r_1^3 r_2^3} \cdot x, \text{ and } \frac{dV}{dy} = m \cdot \frac{x^2+y^2+a^2}{r_1^3 r_2^3} \cdot y,$$

which manifestly prove (a); and from thence, of course

$$\left[\left(\frac{dV}{dx}\right)^2+\left(\frac{dV}{dy}\right)^2\right]^{\frac{1}{2}} = \frac{m}{r_1^3 r_2^3}[x^2+y^2]^{\frac{1}{2}}[(x+a)^2+y^2]^{\frac{1}{2}}[(x-a)^2+y^2]^{\frac{1}{2}},$$

$$= m\frac{r_1 r_2 r}{r_1^3 r_2^3} = \frac{mr}{r_1^2 r_2^2},$$

which manifestly proves (b); and, as (c) follows as an immediate corollary from (a) and (b), by virtue of known elementary properties of the sphere which from their familiarity it is unnecessary to specify, therefore, &c.

5915. (By C. Leudesdorf, M.A.) — The four sides of a rectangle ABCD touch a conic, one of whose foci is S; E, F are points on AB, CD respectively, such that the angles CSE, BSF are right angles; show that AD, EF are equidistant from S.

Solution by R. Tucker, M.A.; C. Swift, B.A.; *and others.*

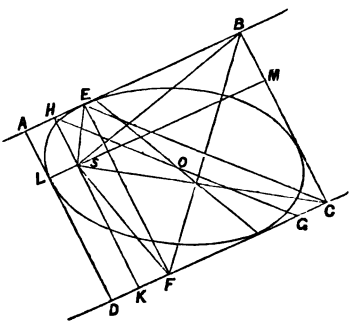
Since S, E, B, C; S, F, C, B are concyclic, therefore a circle will go round SEBCF, and because $\angle ESB = \angle CSF$, therefore $\angle BCE = \angle CBF$, and therefore EF is parallel to AD. Draw HSK, LSM parallel respectively to AD, AB. In the triangles SKF, SBM,

$$\angle FSK = \angle BSM,$$

therefore they are similar; and

$$\frac{BM}{SM} = \frac{KF}{KS},$$

therefore



$KF \cdot SM = BM \cdot KS = b^2 = LS \cdot SM$, therefore $KF = SL$, therefore, &c.

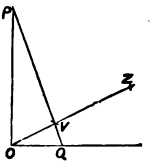
Join H, O, and produce to G, then $HG = 2a$; but $GC = AH = HE$, therefore $CE = 2a$.

5732. (By W. Gallatly, M.A.)—Find, without differentiation, the envelope of $x\cos^3\theta + y\sin^3\theta = a$.

Solution by the Rev. J. L. Kitchin, M.A.; S. Ruggero; *and others.*

Writing $x' = a\cos^3\theta$, $y' = a\sin^3\theta$, we have

$$xx' + yy' = a^2;$$



(x, y') being a point on the 4-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; therefore the line is the polar of (x', y') with regard to circle $x^2 + y^2 = a^2$; therefore the envelope required is the reciprocal polar of the 4-cusped hypocycloid with regard to this circle; therefore, if OV be perpendicular on a tangent to hypocycloid OV, $OZ = a^2$; Z is point on envelope, and $OV = a\sin\theta\cos\theta$; therefore $V\sin\theta\cos\theta = a$ is the envelope.

5864. (By J. Young, B.A.)—Find in what proportion tea at 3*s.* per lb. is mixed with tea at 4*s.* per lb., when by reversing the proportion the quality of the mixture would be improved 10 per cent.

Solution by Morgan Brierley; H. Harley; *and others.*

Let x = the no. of lbs. of tea at 3*s.*; y = the no. of lbs. at 4*s.* When the proportion of tea at 3*s.* to that at 4*s.* is reversed, we have the equation

$$3x + 4y + \frac{3x+4y}{10} = 4x + 3y;\ \text{therefore}\ 2y = x;$$

or 2 lbs. at 4*s.* + 1 lb. at 3*s.* = 11*s.*; and 1 lb. at 4*s.* + 2 lbs. at 3*s.* = 10*s.*, 10 per cent. less.

5898. (By R. A. Graham, M.A.)—A conic is drawn through a point P, and the feet of the normals from it to an ellipse. Show that its centre is the centroid of the points in which a circle of any radius meets the ellipses, the centre of the circle being at P.

Solution by D. Edwardes; G. Turriff; *and others.*

Let (X, Y) be the coordinates of P. The feet of normals are the intersections of ellipse with the conic $a^2yX - b^2Yx - c^2xy = 0$,
and this evidently passes through X, Y. If (x_0, y_0) are the coordinates of its centre, we easily find $x_0 = \frac{a^2X}{c^2}$, $y_0 = -\frac{b^2Y}{c^2}$ ..... (α).

Let the equation of a circle be $(x-X)^2+(y-Y)^2=r^2$.

If θ be eccentric angle of a point of intersection of this circle with ellipse,

$$(a\cos\theta-X)^2+(b\sin\theta-Y)^2=r^2,$$

or $$c^2\cos^2\theta-2aX\cos\theta+\&c.=2Y\sin^2\theta.$$

Squaring each side, the coefficient of $\cos^3\theta$ is easily seen to be $-\frac{4aX}{c^2}$.

Hence $$\tfrac{1}{4}(a\cos\theta_1+a\cos\theta_2+a\cos\theta_3+a\cos\theta_4)=\frac{a^2X}{c^2}=x_0, \text{ by } (a).$$

In the same way we find the coefficient of $\sin^3\theta$ to be $-\frac{4bY}{b^2-a^2}=\frac{4bY}{c^2}$.

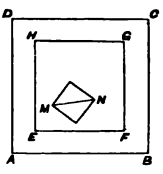
Hence $$\tfrac{1}{4}(b\sin\theta_1+b\sin\theta_2+b\sin\theta_3+b\sin\theta_4)=-\frac{b^2Y}{c^2}=y_0.$$

Hence the truth of the problem.

5674. (By E. B. Seitz, M.A.)—Two points are taken at random in the surface of a square; show that the chance that the square, whose diagonal is the line joining the points, lies wholly within the square, is $\frac{1}{2}$.

Solution by the Proposer.

Let ABCD be the given square, M, N the random points. Let $MN=x$, $AB=a$, and θ equal the angle which NM produced makes with AB. Draw the square EFGH, whose sides are at a distance from those of ABCD equal to $\frac{1}{2}x\cos\theta$; then area EFGH $=(a-x\cos\theta)^2$, which represents the number of favourable relative positions of the two points, while their distance and direction from each other are fixed; and, the conditions remaining the same, the whole number of relative positions of the two points is $(a-x\cos\theta)(a-x\sin\theta)$.



An element of the given square at N is $d\theta\,x\,dx$; also the limits of θ are 0 and $\frac{1}{2}\pi$, and those of x are 0 and $a\sec\theta$. Therefore

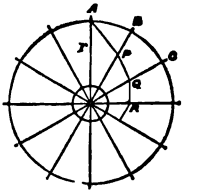
$$p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{a\sec\theta}(a-x\cos\theta)^2\,d\theta\,x\,dx}{\int_0^{\frac{1}{2}\pi}\int_0^{a\sec\theta}(a-x\cos\theta)(a-x\sin\theta)\,d\theta\,x\,dx}=\frac{\int_0^{\frac{1}{2}\pi}\sec^2\theta\,d\theta}{\int_0^{\frac{1}{2}\pi}(2-\tan\theta)\sec^2\theta\,d\theta}=\tfrac{1}{2}.$$

5924. (By Simonelli Ruggero.)—On divise une circonférence en douze parties égales, et l'on joint les points de division au centre. Par

l'un de ces points, on abaisse une perpendiculaire sur le rayon voisin ; du pied de cette perpendiculaire on abaisse une perpendiculaire sur le rayon suivant, et ainsi de suite. On demande de prouver que la limite de la somme de ces perpendiculaires est égale au diamètre, augmenté du côté du triangle équilatéral inscrit.

Solution by T. Mitcheson, B.A., L.C.P.; G. Turriff; *and others.*

Here $AOB = 30° = \theta$ (suppose),



therefore $AP = r \sin\theta$,

$PQ = r \cos\theta \sin\theta$,

$QR = r \cos\theta \cos\theta \sin\theta = r \cos^2\theta \sin\theta$, &c.;

therefore the sum of the perpendiculars

$$= r \sin\theta (1 + \cos\theta + \cos^2\theta + \ldots \text{ to } \infty)$$

$$= \frac{r \sin\theta}{1 - \cos\theta} = r \cot \tfrac{1}{2}\theta = r(2 + \sqrt{3}),$$

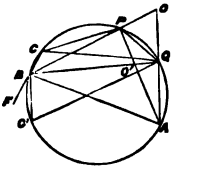
and the side of an inscribed equilateral triangle

$$= 2r \sin\theta r = r\sqrt{3}, \text{ therefore, &c.}$$

5900. (By J. L. McKenzie, B.A.)—BC, BC′ are two fixed equal chords in a given circle, and A is a fixed point in its circumference; PQ is any chord equal to BC or BC′; the lines BP, AQ intersect in O, and BQ, AP in O′; prove that the loci of O and O′ are circles passing through A and B, one of which touches BC, and the other BC′.

Solution by J. C. Glashan, M.A.; E. J. Lawrence, M.A.; *and others.*

1. The angles BPA, PAQ are constant, therefore the angle O is constant; hence the locus of O is a circle passing through A and B.



2. Since PQ = BC′, therefore PB is parallel to QC′, therefore $\angle O = AQC' = ABC'$; hence BC′ is a tangent to the circle AOB.

Also $\angle BO'A = BQA + QAO'$,

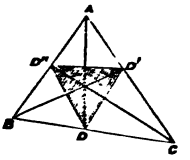
and is therefore constant; hence the locus of O′ is a circle passing through A and B. If CB be produced to F, we have $\angle FBA = APC = BO'A$, because CP and BQ are parallel, therefore CB touches circle about AO′B.

5899. (By Gabriell Torelli.)—On donne un triangle par ses trois côtés, et l'on demande l'expression du rapport de sa surface à celle du triangle qui aurait pour sommets les pieds des bisectrices des angles du triangle donné.

Solution by Professor COCHEZ; D. O'REGAN; *and others.*

Les triangles CDD′, ABC, ayant un angle commun C, sont entre eux comme les côtés qui comprennent cet angle. Donc

$$\frac{CDD'}{ABC} = \frac{CD \cdot CD'}{AC \cdot BC},$$



de même AD′D″ et ABC, BDD″ et ABC donnent respectivement

$$\frac{AD'D''}{ABC} = \frac{AD' \cdot AD''}{AB \cdot AC}, \quad \frac{BDD''}{ABC} = \frac{BD \cdot BD''}{AB \cdot BC}.$$

Si l'on pose BC = a, AC = b, AB = c, et si l'on remarque que D, D′, D″ partagent respectivement les côtés BC, AC et AB en parties proportionelles aux côtés adjacents, on a facilement

$$CD = \frac{ab}{b+c}, \; CD' = \frac{ab}{a+c}, \quad AD' = \frac{bc}{a+c}, \; AD'' = \frac{bc}{a+b},$$

$$BD = \frac{ac}{b+c}, \; BD'' = \frac{ac}{a+b};$$

dès lors

$$\frac{CDD'}{ABC} = \frac{ab}{(b+c)(a+c)}, \quad \frac{BDD''}{ABC} = \frac{ac}{(b+c)(a+b)}, \quad \frac{AD'D''}{ABC} = \frac{bc}{(a+c)(a+b)};$$

d'où $$\frac{\Sigma}{\Delta} = 1 - \frac{ab}{(b+c)(a+c)} - \frac{ac}{(b+c)(a+b)} - \frac{bc}{(a+c)(a+b)};$$

enfin $$\frac{\Sigma}{\Delta} = \frac{2abc}{(a+b)(b+c)(a+c)} = \frac{(a+b)(b+c)(a+c)}{2abc}.$$

[Prof. COCHEZ remarks that his "Δ est l'aire du triangle ABC, Σ l'aire du triangle DD′D;" and he adds furthermore, that "On peut proposer la même question en remplaçant le mot 'bissectrices' par '*hauteurs.*'"]

5680. (By W. J. C. SHARPE, B.A.)—Find (1) the invariant conditions that two conics should have double contact; and (2) the equation to the common tangents in terms of the quantics and their invariants.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

For double contact, the equation to the chords of intersection

$$\Delta S'^3 - \theta S'^2 S + \theta' S' S^2 - \Delta' S^3 = 0 \text{ (SALMON'S } \textit{Conics}\text{, p. 322)}$$

must have a pair of equal roots denoting the common chord $S' - S$; therefore the above equation must be equivalent to

$$(S' - S)^2 (\Delta' S - \Delta S') = 0; \text{ and therefore } \theta = \Delta' + 2\Delta, \; \theta' = 2\Delta' + \Delta,$$

which are equivalent to $\theta - \theta' = \Delta - \Delta'$ and $\theta + \theta' = 3(\Delta + \Delta')$; and the equation to the common tangents is $\Delta S' - \Delta' S = 0$.

5493. (By Professor SYLVESTER, F.R.S.)—If χ_q represent in general the number of linearly independent covariants of the degree q in the variables, and of a given order j in the constants belonging to a binary quantic of the degree i, prove that

$$\chi_0 + 2\chi_1 + 3\chi_2 + 4\chi_3 + 5\chi_4 + \ldots\ldots = \frac{\Pi(i+j)}{\Pi i \,.\, \Pi j}.$$

Note.—Of course χ_0 represents the number of invariants, and the series stops spontaneously after the $(ij+1)^{\text{th}}$ term by all the subsequent χ's vanishing. Of course, when ij is even, all the odd-indiced χ's, and when ij is odd, all the even-indiced χ's, vanish. The theorem as stated comprehends both of these cases in one.

Note by the PROPOSER.

This is a grand theorem which occurs as a corollary to my proof of the *Fundamental Theorem* in Invariants, namely, of the linear independence of a certain general system of derived functions which was established by experience a quarter of a century ago, but had never been demonstrated. I told a pupil at Johns Hopkins University not long ago that, although the fact of this independence was perfectly certain, "it transcended, I thought, the powers of the human understanding to give a proof of it." That proof occurred to me the day before yesterday quite suddenly, like a ray of light entering my brain, and will be sent to CRELLE'S *Journal*. It is, I think, the most wonderful kind of proof ever given of an algebraical fact.

5921. (By R. A. ROBERTS, M.A.)—In the equilibrium of a uniform cord stretched over a smooth sphere and repelled according to the law of the inverse cube of the distance by a uniform circular ring lying on the sphere, if the tension at any point be that to infinity under the action of the force, prove that the catenary will be a circle orthogonal to the ring.

Solution by Professor TOWNSEND, F.R.S.

This pretty extension to a sphere of a known property of two uniform orthogonal circles in a plane, noticed I believe first by myself, may be readily reduced to its analogue in the plane, as follows:—

It may be easily shewn, without any difficulty whatever, that the repulsion of the ring, at any point on the sphere, passes in direction through the harmonic pole of its plane with respect to the surface, and varies in magnitude directly as the distance from the pole and inversely as the square of the distance from its plane. Hence, for any circle of the sphere whose plane passes through the pole, and whose circumference consequently intersects at right angles that of the ring, the repulsion varies directly as the distance from that point and inversely as the square of the distance from its polar with respect to the circle. But that is precisely the law of central force requisite to the free equilibrium of a uniform cord in a circular catenary, for which the tension of maintenance

is, as is well known, that to infinity under the action of the force; and therefore, &c.

N.B.—As the ring, in the above, may manifestly be flexible as well as the cord, and both perfectly free in the space of the sphere, as far as the equilibrium of either under the repulsive action of the other for the law of force is concerned, the entire action at every point of each lying in its plane and over the entire circumference of each making equilibrium in its plane; and, as any two circles in a common space which intersect each other twice in the space lie manifestly on a common sphere in the space, it follows consequently from the above, as originally stated by myself for the case of a common plane, that *two uniform circular cords, intersecting twice at right angles in a common space, and repelling each other according to the law of the inverse cube of the distance, will keep each other in free equilibrium in the space.*

5307. (By Professor WOLSTENHOLME, M.A.)—If S_r denote the infinite series $1^{r-1}+2^{r-1}+\frac{3^{r-1}}{2!}+\ldots+\frac{(n+1)^{r-1}}{n!}+\ldots$ to ∞,

prove that $\frac{S_r}{S_1}=\Delta 0^r+\frac{\Delta^2 0^r}{2!}+\frac{\Delta^3 0^r}{3!}+\ldots+\frac{\Delta^r 0^r}{r!}$, a finite series.

Solution by E. B. ELLIOTT, M.A.; H. STABENOW, M.A.; *and others.*

Here $S_r=0^r+1^r+\frac{2^r}{2!}+\frac{3^r}{3!}+\ldots$

$$=\epsilon^E 0^r=\epsilon\,.\,\epsilon^\Delta 0^r=\epsilon\left\{1+\Delta+\frac{\Delta^2}{2!}+\frac{\Delta^3}{3!}+\ldots\right\}0^r;$$

also $S_1=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots=\epsilon$;

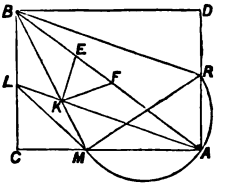
therefore $\frac{S_r}{S_1}=\left\{1+\Delta+\frac{\Delta^2}{2!}+\frac{\Delta^3}{3!}+\ldots\right\}0^r$

$$=\Delta 0^r+\frac{\Delta^2 0^r}{2!}+\frac{\Delta^3 0^r}{3!}+\ldots+\frac{\Delta^r 0^r}{r!},$$

the number of terms being finite as stated, for $\Delta^m 0^r$ vanishes if m be greater than r. [See Question 1653 of the second edition of Prof. WOLSTENHOLME's admirable collection of Mathematical Problems.]

5885. (By the EDITOR.)—Determine a right-angled triangle, having given the two lines bisecting the acute angles, and terminated by the opposite sides; and show that, if the lengths of the bisectors be 40 and 50, the lengths of the sides will be 35·80738, 47·10728, 59·41143.

I. *Solution by* R. TUCKER, M.A.



Take a rectangle ACBD; and let AL, BM, the bisectors of A, B, intersect in K; then $\angle AKB = 135°$. Through B draw BR parallel to AL to meet AD in R, then BR = AL. Hence, from data, the triangle BMR is known.

It is well known that

$2\triangle BMR + CM \cdot DR$

$= \text{rect. } AB = 2\triangle BMR + 2\triangle CLM$ .....(1).

Take BE = BL, AF = AM, then

$$\triangle BKE = \triangle BKL,\ \triangle AKF = \triangle AKM, \text{ and } \triangle FKE = \triangle LKM,$$

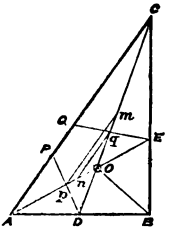
because the angles LKM, FKE are supplementary; therefore quadrangle $AMLB = 2\triangle AKB$; hence, by (1), $\triangle AKB = \frac{1}{4}\triangle BMR$.

Construction.—Make a triangle BMR, having its sides BM, BR equal to the given bisectors, and the angle MBR equal to half a right angle. On MR draw a semicircle; and construct an hyperbola having BM, BR, for asymptotes, and such that the rectangle under the ordinate and abscissa (parallel to the asymptotes) is half the rectangle under the given bisectors.

Let this hyperbola cut the semicircle in A; join AB, and produce AK, parallel to BR, so that AL = BR, and produce BL, AM to meet in C.

Then ABC will be the triangle required.

II. *Solution by the* PROPOSER.



1. Let ABC be the triangle; AE $(= 2a = 40)$, CD $(= 2b = 50)$ the bisectors, which meet in O; DpP, EqQ perpendiculars to AE, CD respectively; and pm, qn parallels to AC. Then DpP is bisected in p, and EqQ in q; consequently, CD is bisected in m, and AE in n.

And since we have

$$\angle COE = \tfrac{1}{2}\angle A + \tfrac{1}{2}\angle C = 45° = \angle DOA = \angle CBO,$$

therefore $Op = \dfrac{OD}{\sqrt{2}}$, and $Oq = \dfrac{OE}{\sqrt{2}}$.

Let $On = x$, and $Om = y$, then we have

$$\frac{OA}{OC} = \frac{Op}{Om} = \frac{On}{Oq}, \text{ or } \frac{a+x}{b+y} = \frac{b-y}{\sqrt{2}.y} = \frac{\sqrt{2}.x}{a-x};$$

whence $\dfrac{x}{a} = \dfrac{b-y}{b+y}$; and $\left(\dfrac{y}{b}\right)^3 + \left(\dfrac{y}{b}\right)^2 + \dfrac{2\sqrt{2} : a-b}{b}\left(\dfrac{y}{b}\right) = 1$ ...(a).

Having thus found x and y, we have

$$AC = \left\{(a+x)^2 + \sqrt{2}\,(a+x)\,(b+y) + (b+y)^2\right\}^{\frac{1}{2}};$$

$$AB = \frac{AO^2}{AD} = \frac{(a+x)^2}{\left\{(a+x)^2 - \sqrt{2}\,(a+x)\,(b-y) + (b-y)^2\right\}^{\frac{1}{2}}}.$$

2. Otherwise:—Put $\angle BAC = 2\theta$, then we have

$$a\cos\theta\tan 2\theta = b\cos(45° - \theta);$$

or $$\tan^2\theta + \tan^2\theta + \frac{2\sqrt{2}\,.\,a-b}{b}\tan\theta = 1 \quad \ldots\ldots\ldots\ldots(\beta),$$

an equation wherefrom all the parts of the triangle may be readily determined.

3. Or, again, for a third solution, let

$$AE = a = 40,\ CD = b = 50,\ BC = x,\ AB = y,\ AC = z;$$

then we have $x^2+y^2=z^2$, or $$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1 \quad \ldots\ldots\ldots\ldots(1).$$

Now, since $$AC + CB : CB = AB : BD = \frac{xy}{x+z},$$

therefore $$b^2 = x^2 + \frac{x^2y^2}{(x+z)^2};$$

therefore $b^2(x+z)^2 = x^2\{y^2+(x+z)^2\} = 2x^2z(x+z)$, by (1);

whence we obtain $$b^2(x+z) = 2x^2z \quad \ldots\ldots\ldots\ldots(2);$$

also, by similarity, $$a^2(y+z) = 2y^2z \quad \ldots\ldots\ldots\ldots(3).$$

Equation (2)÷(3) gives $$\frac{b^2}{a^2}\cdot\frac{x+z}{y+z} = \frac{b^2}{a^2}\cdot\frac{\frac{x}{z}+1}{\frac{y}{z}+1} = \frac{x^2}{y^2} \quad \ldots\ldots\ldots\ldots(4).$$

Assume $\frac{x}{z} = \frac{2v}{1+v^2}$, then (1) gives $\frac{y}{z} = \frac{1-v^2}{1+v^2}$, and $\frac{x}{y} = \frac{2v}{1-v^2}$.

Substituting these values in (4), we have

$$\frac{b}{a}\cdot\frac{1+v}{\sqrt{2}} = \frac{2v}{1-v^2},$$

or $$v^3+v^2+\frac{2\sqrt{2}\,.\,a-b}{b}v = 1 \quad \ldots\ldots\ldots\ldots(\gamma).$$

With the values of a and b given above, equation (γ) becomes

$$v^3+v^2+1{\cdot}26274169v = 1,$$

whence we obtain

$$v = {\cdot}497881647,\ \frac{x}{z} = \frac{2v}{1+v^2} = {\cdot}79796006,\ \frac{y}{z} = \frac{1-v^2}{1+v^2} = {\cdot}60271033.$$

By (2) and (3), since $a = 40$, $b = 50$, we have

$$x^2 = 1250\left(\frac{x}{z}+1\right) = 2247{\cdot}45008,\ y^2 = 800\left(\frac{y}{z}+1\right) = 1282{\cdot}168264;$$

and from these values we obtain, at once,

$$x = 47{\cdot}407279 = BC,\ y = 35{\cdot}807377 = AB,\ z = 59{\cdot}41143 = AC.$$

It will be observed that the equations (α), (β), (γ) are the same in the three solutions.

5881. (By Professor Darboux.)—Sur un cercle donné O, on prend à volonté un arc ANB, et, sur la corde AB de cet arc, on décrit une demi-

circonférence AMB, puis on fait tourner la figure autour du diamètre perpendiculaire à AB. On demande quelle doit être cette corde pour que la somme des surfaces décrites par les lignes AMB, ANB, soit maximum.

Solution by E. ANTHONY, M.A.; W. J. MACDONALD, M.A.; *and others.*

If C be the middle point of the chord AB, a the radius of the circle, and $OC=y$, we have the surfaces traced out by the semicircle BMA and the arc BNA equal, respectively, to

$$2\pi(a^2-y^2), \text{ and } 2\pi a\int_{-y}^{a} dx \text{ or } 2\pi a(a+y);$$

therefore sum of the surfaces $= 2\pi(a+y)(2a-y)$.

And this is a maximum when $y=\frac{1}{2}a$; that is, when the chord AB $= a\sqrt{3}$.

5897. (By C. LEUDESDORF, M.A.)—Show that, if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$,

$$\left(\frac{a^2-bc}{b+c}+\frac{b^2-ca}{c+a}+\frac{c^2-ab}{a+b}\right)\left(\frac{b+c}{a^2-bc}+\frac{c+a}{b^2-ca}+\frac{a+b}{c^2-ab}\right)=3\frac{a^3+b^3+c^3}{abc}.$$

Solution by the Rev. D. THOMAS, M.A.; A. W. SCOTT, M.A.; *and others.*

If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0=\Sigma\frac{1}{a}$, $bc=-a(b+c)$, $a^2-bc=a.\Sigma a$;

$$\therefore\quad \Sigma\frac{a^2-bc}{b+c}\Sigma\frac{b+c}{a^2-bc}=\Sigma\frac{a}{b+c}\Sigma\frac{b+c}{a}=-\Sigma\frac{a^3}{abc}\left\{\Sigma a.\Sigma\frac{1}{a}-3\right\}=3\frac{\Sigma a^3}{abc}.$$

5912. (By J. J. WALKER, M.A.)—Prove that $\mathrm{V}.\mathrm{V}\beta\gamma\mathrm{V}\gamma\alpha$ represents the moment of force β, acting at the end of the vector α, about the (unit) axis γ.

Solution by G. TURRIFF, M.A.; W. J. MACDONALD, M.A.; *and others.*

Now $\mathrm{V}.\mathrm{V}\beta\gamma\mathrm{V}\gamma\alpha=\alpha\mathrm{S}(\mathrm{V}\beta\gamma.\gamma)-\gamma\mathrm{S}(\alpha\mathrm{V}\beta\gamma)=-\gamma\mathrm{S}\alpha\beta\gamma$;

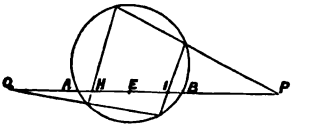
so that the expression represents a vector which represents the moment in direction. Also, if θ be the angle between α and β, and ϕ the angle between their plane and γ, we have (TAIT'S *Quaternions*, p. 64)

$$\mathrm{S}\alpha\beta\gamma=-\mathrm{T}\alpha\mathrm{T}\beta\mathrm{T}\gamma\sin\theta\cos\phi=-\mathrm{T}\alpha\mathrm{T}\beta\sin\theta\cos\phi,$$

which represents the magnitude of the moment; consequently $\mathrm{V}.\mathrm{V}\beta\gamma\mathrm{V}\gamma\alpha$ represents the moment in magnitude and direction.

5634. (By CHRISTINE LADD.)—A secant, bisected in E, meets the successive sides of a quadrilateral inscribed in a circle in the points P, I, Q, H respectively; show that, if EI = EH, EP = EQ.

Solution by the PROPOSER.



If the secant meet the circle in the points A, B, we have, by a well-known theorem,

$$HA \cdot HB \cdot IP \cdot IQ = IA \cdot IB \cdot HP \cdot HQ \ldots (m).$$

But $HA \cdot HB = \overline{AE}^2 - \overline{HE}^2$,

$IP \cdot IQ = (PE - EI)(QE + EI)$, $IA \cdot IB = \overline{AE}^2 - \overline{IE}^2$,

$HP \cdot HQ = (PE + HE)(QE - HE)$.

Substituting these values in (m), we have

$$(PE - QE)(\overline{AE}^2 - IE \cdot HE) = (IE - HE)(\overline{AE}^2 - PE \cdot QE).$$

If $IE - HE = 0$, then $(PE - QE)(\overline{AE}^2 - IE \cdot HE) = 0$; but, since $\overline{AE}^2$ is not equal to $\overline{IE}^2$, this must be equivalent to PE = QE.

5922. (By R. E. RILEY, B.A.)—Tangents are drawn to an ellipse from points in a concentric circle, radius m; show that the middle points of the chords of contact lie on the curve $r(a^2 \sin^2\theta + b^2 \cos^2\theta) = a^2b^2m^{-1}$.

Solution by C. SHARP, B.A.; R. KNOWLES, L.C.P.; *and others.*

Take the common centre O as origin, and let P (m, ϕ) be a point on the circle; then the equation to a chord of contact of tangents drawn from P is

$$mr(b^2 \cos\theta \cos\phi + a^2 \sin\theta \sin\phi) = a^2b^2;$$

hence, since OP bisects the chord of contact, by putting $\phi = \theta$, we obtain

$r(b^2 \cos^2\theta + a^2 \sin^2\theta) = a^2b^2m^{-1}$, the required locus.

5790. (By E. W. SYMONS.)—Prove that the equation of the tangent to an involute of $u = 0$, at the point corresponding to (xy), is

$$(X - x)\frac{du}{dy} - (Y - y)\frac{du}{dx} = s\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)^{\frac{1}{2}},$$

s being the length of the arc of u measured from a certain fixed point on the curve to (x, y).

Solution by W. T. CURRAN SHARP, B.A.; *the* PROPOSER; *and others.*

The tangent at any point of the involute is parallel to the normal at the corresponding point of the curve, since both are perpendicular to the tangent to the curve at this point. Hence the equations to the tangent to the involute, and the perpendicular thereupon from (x, y), are

$$(X-x)\frac{du}{dy} - (Y-y)\frac{du}{dx} = p, \quad p\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right\}^{-\frac{1}{2}},$$

and the latter must be equal to the length of the tangent to the curve at (x, y), that is, by definition, of involute to S; therefore

$$p = s\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2\right\}^{\frac{1}{2}},$$

and the equation to the tangent to the involute is as in the Question.

5664. (By Professor TAIT, F.R.S.)—Show that the greatest amount of mechanical effect which can be obtained from a system of equal and similar masses (whose specific heat does not vary with their temperature) is proportional to the excess of the arithmetic over the geometric mean of their absolute temperatures.

Solution by J. J. WALKER, M.A.

The deduction of this particular case from the equations applicable to a more general system has been made by Professor TAIT himself in his *Thermodynamics*, 2nd ed. § 181; but the object in proposing it as above appearing to be to elicit a simple proof specially applicable to this case, the following may be given.

Using Prof. TAIT's notation, let m be the common mass, t, t' the absolute temperatures, c the specific heat, while T is the absolute temperature to which the system is brought, so that all the heat lost is converted into work. Then $W = Jcm[t-T-(T-t')] = Jcm(t+t'-2T)$; but $\frac{t}{T} = \frac{T}{t'}$, from the definition of absolute temperature, or $T = (tt')^{\frac{1}{2}}$.

Hence finally $W = 2Jcm[\frac{1}{2}(t+t') - (tt')^{\frac{1}{2}}]$.

5545. (By J. C. MALET, M.A.)—Prove that the locus of the centre of a circle which cuts orthogonally a given circle, and touches a given conic, is a curve of the sixth order, which has six cusps that lie all on another conic; and hence prove that the six cusps of the first negative pedal of a central conic with respect to any point all lie on another conic.

Solution by J. W. SHARPE, M.A.; E. RUTTER; *and others.*

Take the equation to the conic under the form $\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1=0$, and the equation to the circle under the form

$$\xi^2+\eta^2-2u\xi-2v\eta-w^2=0.$$

Then the condition that this circle be cut orthogonally by the variable circle $(\xi-x)^2+(\eta-y)^2-r^2=0$ is

$$2ux+2vy-x^2-y^2+r^2+w^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Calculating the invariants of the conic and the variable circle, we get

$$\Delta=-\frac{1}{a^2b^2}, \quad \Delta'=-r^2, \quad \Theta=\frac{x^2+y^2-r^2-a^2-b^2}{a^2b^2},$$

$$\Theta'=\frac{b^2x^2+a^2y^2-a^2b^2-r^2(a^2+b^2)}{a^2b^2}.$$

We shall express the condition that the variable circle touch the conic; and, eliminating r^2 between the resulting equation and the equation (α) above, we shall obtain the equation to the locus of the point (x, y), which is the centre of the variable circle. Now put

$$L=2ux+2vy+w^2-a^2-b^2, \quad C=r^2 \text{ or } x^2+y^2-L-a^2-b^2,$$

$$\Sigma=b^2x^2+a^2y^2-a^2b^2-(a^2+b^2)C;$$

so that $C=0$ is the equation to the given circle; and $\Sigma=0$ is a conic passing through the four points common to the given circle and conic; and $L=0$ is the equation to a straight line. Then

$$\Delta=-\frac{1}{a^2b^2}, \quad \Delta'=-C, \quad \Theta=\frac{L}{a^2b^2}, \quad \Theta'=\frac{\Sigma}{a^2b^2}.$$

Now the condition that the variable circle should touch the conic is

$$(\Theta\Theta'-9\Delta\Delta')^2=4(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta),$$

which becomes upon substitution

$$(L\Sigma-9a^2b^2C)^2=4(L^2+3\Sigma)(\Sigma^2+3a^2b^2CL).$$

This equation remains of precisely the same form, whatever be the form in which we assume the equation to the given conic. For the coefficients of the conic enter into L under the form a^2+b^2, and into the Σ under the forms a^2+b^2 and a^2b^2; and these are both immediately expressible in terms of the two invariants of the given conic. Hence we have obtained the general equation to the locus of the centre of the variable circle.

The form of the equation shows it to be of the sixth degree. Also it touches the conic $L^2+3\Sigma=0$ in the six points in which the conic is intersected by the cubic $L\Sigma-9a^2b^2C=0$, and touches the quartic $\Sigma^2+3a^2b^2CL=0$ in the twelve points where it is cut by this cubic. Now, if we eliminate C between the last two equations, we get the complex $\Sigma(L^2+3\Sigma)=0$; hence the last twelve points are the intersections of the cubic with the two conics $\Sigma=0$, and $L^2+3\Sigma=0$. Hence it appears that the six points of intersection of the cubic with the latter conic are points where the locus touches both the conic and the quartic; hence these six points are double points of some kind or other. We shall now show that they are cusps.

The equation to the locus becomes, when expanded,

$$L^2\Sigma^2+18a^4L\Sigma C-27a^8C^2+4a^4L^3C+4\Sigma^3=0,$$

where we have put a^4 for a^2b^2.

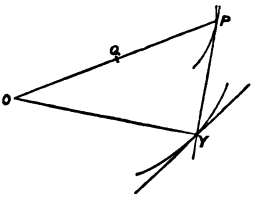
Now $$(L^2+3\Sigma)^3=L^6+9L^4\Sigma+27L^2\Sigma^2+27\Sigma^3.$$

Whence it immediately appears that the equation to the locus required may be written in the form $(2L^3+9L\Sigma-27a^4C)^2-4(L^2+3\Sigma)^3=0$, which shows that the six points, in which the cubic $2L^3+9L\Sigma-27a^4C=0$ cuts the conic $L^2+3\Sigma=0$, are cusps. That these are the six double points above discussed becomes evident upon combining these two equations.

The tangents at the cusps are of course the tangents at those points to the cubic.

For the second part of the problem let the given circle reduce to a point circle. The variable circle must then pass through this point, because it cuts the given circle in all cases orthogonally.

Let O be this given point; P the point on the negative pedal which corresponds to the point Y on the conic, so that PY is the tangent to the pedal at P. Then the tangent at Y to the conic and that to the pedal at P are equally inclined to OY and OP respectively; hence the circle described about the triangle OPY touches the conic at Y; also, OYP being a right angle, its centre is Q, the middle point of OP; hence the locus P is similar to the locus of Q. But by the above investigation the latter has six cusps lying upon a conic; therefore so also has the locus of P. Hence the required proposition is proved.



5892. (By the Rev. W. A. Whitworth, M.A.)—If C_r^n denote the number of combinations of n things r together, prove that

$$\Sigma_{\alpha=0}^{\alpha=a}\Sigma_{\beta=0}^{\beta=b}\Sigma_{\gamma=0}^{\gamma=c}\cdots\Sigma_{\mu=0}^{\mu=m}\frac{C_\alpha^a C_\beta^b C_\gamma^c\ldots C_\mu^m}{C_{\alpha+\beta+\gamma+\ldots+\mu}^{a+b+c+\ldots+m}}=1+a+b+c+\ldots+m.$$

5913. (By the Rev. W. A. Whitworth, M.A.)—If C_r^n denote the number of combinations of n things r together, prove that, whatever be the value of ν,

$$\Sigma_{\alpha=0}^{\alpha=a}\Sigma_{\beta=0}^{\beta=b}\Sigma_{\gamma=0}^{\gamma=c}\cdots\Sigma_{\mu=0}^{\mu=m}\frac{C_\alpha^a C_\beta^b C_\gamma^c\ldots C_\mu^m C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}}=\frac{1+a+b+c+\ldots+m+n}{1+n}$$

Solution of both Questions by the Proposer.

Consider a parallelepiped whose edges are m, n, r, and divide it by lines parallel to its edges into mnr parallelepipeds, all of whose edges are unity.

Let it be required to go from one corner to the opposite corner of the great parallelepiped by traversing edges of the little parallelepipeds without retrogression.

The number of ways is evidently $\frac{(m+n+r)!}{m!\,n!\,r!}$, for which we may conveniently write $C_{m,n,r}$.

Classify these routes according to the ways in which they severally pass from the plane $Z = z$ to the plane $Z = z+1$, regarding the initial point as origin and the three edges through this point as axes of coordinates.

The number of routes passing from the point (x, y, z) to the point $(x, y, z+1)$ is $\quad C_{x,y,z} \cdot C_{m-x,\, n-y,\, r-z-1}$.

Give all possible values to x and y and the summation must give all the routes, therefore

$$\Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} C_{x,y,z} \cdot C_{m-x,\, n-y,\, r-z-1} = C_{m,n,r}$$

$$\text{or} \quad 1 = \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{C_{x,y,z}\, C_{m-x,\, n-y,\, r-z-1}}{C_{m,n,r}}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{(x+y+z)!\,(m+n+r-x-y-z-1)!\, m!\, n!\, r!}{x!\, y!\, z!\, (m-x)!\, (n-y)!\, (r-z-1)!\, (m+n+r)!}$$

$$= \Sigma_{x=0}^{x=m} \Sigma_{y=0}^{y=n} \frac{\dfrac{m!}{x!\,(m-x)!} \dfrac{n!}{y!\,(n-y)!} \dfrac{(r-1)!}{z!\,(r-z-1)!}}{(x+y+z)!\,(m+n+r-x-y-z-1)!} \cdot \frac{r}{m+n+r}$$

Now write α, β, γ for x, y, z, and a, b, c for $m, n, r-1$. Then our equation becomes $\dfrac{a+b+c+1}{c+1} = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \dfrac{C_\alpha^a\, C_\beta^b\, C_\gamma^c}{C_{\alpha+\beta+\gamma}^{a+b+c}}$,

which is a particular case of Question 5943.

Now imagine space of N dimensions and make the same construction as before; then, if the N coordinates be $\alpha, \beta, \gamma \ldots \mu, \nu$, and the corresponding constants $a, b, c \ldots m, n$, our result becomes

$$\frac{a+b+c+\ldots+m+n+1}{n+1} = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \cdots \Sigma_{\mu=0}^{\mu=m} \frac{C_\alpha^a\, C_\beta^b\, C_\gamma^c \ldots C_\mu^m\, C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}},$$

which is the theorem of Question 5943.

The sum $\dfrac{a+b+c+\ldots+m+n+1}{n+1}$ is independent of ν and will therefore be the same for all values of ν. Give ν all the values $0, 1, 2, \ldots n$ in succession and add. Then we get

$$a+b+c+\ldots+m+n+1 = \Sigma_{\alpha=0}^{\alpha=a} \Sigma_{\beta=0}^{\beta=b} \cdots \Sigma_{\mu=0}^{\mu=m} \Sigma_{\nu=0}^{\nu=n} \frac{C_\alpha^a\, C_\beta^b\, C_\gamma^c \ldots C_\mu^m\, C_\nu^n}{C_{\alpha+\beta+\gamma+\ldots+\mu+\nu}^{a+b+c+\ldots+m+n}},$$

which is the theorem of Question 5892, with n written for m.

5814. (By H. L. Orchard, L.C.P.)—Two spherical soap-bubbles are blown, the one from water, the other from a mixture of water and alcohol. If the unit superficial tensions per linear inch be 1 grain and $\frac{7}{13}$ grain respectively, and the radii be $\frac{7}{8}$ inch and $1\frac{1}{8}$ inch respectively, compare, in the two cases, the excess of the total internal pressure over the external pressure.

5944. (By H. L. Orchard, L.C.P., &c.)—A spherical soap-bubble is electrified in such a manner that the internal pressure p remains constant. Show that, if ρ_n, ρ_{n+1}, respectively, are the electrical densities when its volume has become n and $n+1$ times its initial volume, we must have the relation $\rho_{n+1} : \rho_n = n : (n^2-1)^{\frac{1}{2}}$.

Solution of both Questions by J. J. Walker, M.A.

The solution of the former of these questions is obvious, as from the well-known expression $p = \frac{2T}{r}$, giving for the total pressure $Sp = 8\pi Tr$, whenever, as in the example proposed, the tensions are inversely as the diameters, the total pressures will be equal.

The second, though not practically verifiable, is ingenious; and such questions have an educational value. Let a be the initial radius, then the work done in enlarging the sphere to radius r by a uniform pressure p, will be equal to $4\pi p \int_a r^2\,dr = \frac{4}{3}\pi p\,(r^3-a^3)$, which, when $r^3 = na^3$, reduces to $\frac{4}{3}\pi p\,(n-1)\,a^3$, and when $r^3 = (n+1)\,a^3$, to $\frac{4}{3}\pi p\,na^3$. Equating these to the potential energies of the electrifications, *i. e.*, to $\frac{Q^2}{2r}$, half the product of charge and potential, according to Helmholtz's formula, or $8\pi^2\rho^2 r^3$, in each case—ρ being the electrical density—there results

$$(n+1)\,\rho_{n+1}^2 : n\rho_n^2 = n : n-1 \text{ or } \rho_{n+1}^2 : \rho_n^2 = n^2 : n^2-1.$$

5742. (By Frederick Purser, M.A.)—A conic U touches a line L and a conic V, and has double contact with another conic W; show that the chord of contact of U and W envelopes a binodal quartic of which the points of intersection of L and W are the nodes.

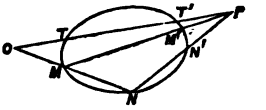
Solution by the Proposer.

1. The bicircular quartic is generated as the envelope of a circle whose centre lies on a conic F and which cuts a circle J orthogonally. Hence follows readily the following generation of the curve as a locus. Let a line parallel to the polar with respect to a circle of any point P, and bisecting the interval between P and the polar, be termed the quasi-polar of P with respect to the circle; then the bicircular quartic is the locus of points whose quasi-polars with respect to the circle J touch the conic F.

If now the two circular points at infinity be replaced by any two points M, N, the circle J becomes a conic through MN, and the quasi-polar of any point P is now the chord joining the other intersections with J of the

lines PM, PN; the binodal quartic being then the locus of points whose quasi-polars to J touch F.

2. Now it may be shown as follows, that the quasi-polars with respect to J of points on a given right line envelope a conic having double contact with J and touching the line MN.



For, let M', N', T, T', be the conic J; T, T' the given line; P any point on it; M'N' its quasi-polar. Then, as P moves along TT', the points M', N' trace out homographic systems, and therefore M'N' envelopes a conic U having double contact with J. When P is at T, T', the chord M'N' becomes the tangent to J at T, T', showing that the given line TT' is the chord of contact of J with the envelope conic. Lastly, let P be at O the intersections of MN, TT', then M'N' coincides with MN. Hence MN is a line of the enveloping system, or the envelope conic U touches MN.

3. Let it now be required to find the points in which a given line TT' meets the binodal quartic. From what has been proved, it appears that the quasi-polars of these points are the four common tangents to the conic F and the envelope conic U. The four corresponding poles on TT' are then the points required. Let now the envelope conic U touch F; then two of these points will coincide, or the line TT' will touch the binodal quartic. Hence any tangent TT' to the binodal quartic is the chord of contact with J of a conic U touching the line MN, the conic F, and having double contact with J.

4. I would add that the same method leads to the more general theorem, "A conic U touches a line L, has double contact with a conic W, and touches a given curve of the n^{th} class. The envelope of the chord of contact of U and W is a curve of degree $2n$, having the two points of intersections of L and W as multiple points of order n.

5813. (By W. R. Roberts, M.A.)—From any point on an inflexional tangent to a cubic can be drawn four tangents to the curve. Show that there are six points, two on each inflexional tangent lying on a conic, from which the tangents form an harmonic pencil.

Solution by J. J. Walker, M.A.

It seems as if this theorem had been obtained by transformation for a nodal cubic, and it had been inferred that it must be true for non-singular cubics also; but, on testing this generalisation, it appears not to be justified.

Let a cubic referred to the three inflexion tangents be

$$(x+y+z)^3+3mxyz=0;$$

then, if from $(x'y')$, a point on $z=0$, tangents are drawn to the curve, I find

for the condition that they should form a harmonic pencil, if $w' = x' + y'$,

$$27w'^{12} + 54mw'^{10}x'y' + 45m^2w'^8x'^2y'^2 - 180m^2w'^6x'^3y'^3$$
$$-15(12m^3+m^4)w'^4x'^4y'^4 - 6(10m^4+m^5)w'^2x'^5y'^5$$
$$-(24m^4+12m^5+m^6)x'^6y'^6 = 0,$$

which, on examination, will not, I think, be found to have a quadratic factor; but if the condition for the curve being nodal be introduced—viz., $m = -9$,—then the above condition becomes $27(w'^2-3x'y')^6 = 0$, and there are, accordingly, two points on each tangent possessing the property. The six points evidently lie on the conic

$$x^2+y^2+z^2-yz-zx-xy = 0.$$

I have checked the general result above by calculating it independently for the ranges made on the lines $y = 0$ and $x+y+z = 0$; and its reducing as it does, for $m = -9$ is a further test of its correctness.

5888. (By R. Tucker, M.A.)—D, E, F are the points of contact of an inscribed circle with the sides. Circles are inscribed in AFE, BFD, CDE, and their centres are joined. The triangle A′B′C′ thus formed is treated in the same way as the triangle ABC, and so on: prove that the triangle $A_\infty B_\infty C_\infty$ is equilateral.

Solution by J. O'Regan; *the* Proposer; *and others.*

It is readily seen that A′, B′, C′ are the middle points of the arcs EA′F, FB′D, DC′F. Hence $\angle$A′, which stands upon B′DC′, is half angle on FDE, *i.e.*, $= 45° + \frac{1}{4}A$, therefore $A'-B' = \frac{1}{4}(A-B)$; and similarly

$$A_n - B_n = \frac{A-B}{4^n}, \text{ and } A_\infty = B_\infty = C_\infty,$$

which proves the property.

5937. (By Septimus Tebay, B.A.)—Show that, to the first order of approximation, the equation to the section of a comet (consisting of a small spherical nucleus, surrounded by light nebulous matter) by the plane of its orbit is $r = a - b\cos\theta + c\cos 2\theta$; the equilibrium being statical,—an assumption suggested by the general apparition of all comets.

Solution by the Proposer.

Let a be the radius of the undisturbed mass of the comet, μ the mass of the nucleus, m the mass of the sun, d the distance between their centres, r the distance of any point (xyz) of the nebula, r' its distance from the

sun, x being measured in the direction of d, and w the angular velocity about the sun.

Since the angular velocity has no effect perpendicular to the plane of the orbit, we have

$$X = -\frac{\mu}{r^3}x + \frac{m}{r'^3}(d-x) - \frac{m}{d^2} - w^2(d-x),$$

$$Y = -\frac{\mu}{r^3}y - \frac{m}{r'^3}y + w^2y, \quad Z = -\frac{\mu}{r^3}z - \frac{m}{r'^3}z.$$

Hence the equation to the free surface is

$$\frac{\mu}{r} + \frac{m}{r'} - \frac{mx}{d^2} + \tfrac{1}{2}w^2[(d-x)^2 + y^2] = \text{const.}$$

Considering only the section of the comet by the plane of its orbit, and putting $x = r\cos\theta$, $y = r\sin\theta$, $r = a + \eta$, η being a small quantity having no mean value, we find

$$\eta = -daw^2\cos\theta + \frac{3}{4}\frac{m}{\mu}\left(\frac{a}{d}\right)^2 a\cos 2\theta;$$

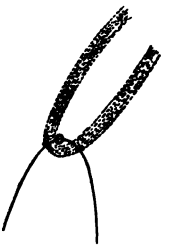
and therefore

$$r = a\left\{1 - dw^2\cos\theta + \frac{3}{4}\frac{m}{\mu}\left(\frac{a}{d}\right)^3\cos 2\theta\right\}.$$

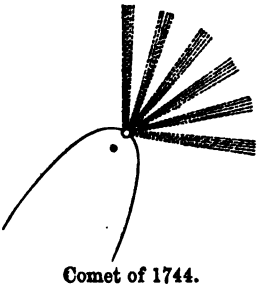
The length of the tail is therefore

$$a\left\{dw^2 + \frac{3}{4}\frac{m}{\mu}\left(\frac{a}{d}\right)^3\right\}.$$

As d diminishes, the solar heat increases; it is therefore probable that a also increases, through the expansion of the nebulous matter of the comet. Also, since the angular velocity has no effect perpendicular to the plane of the orbit, there will be a consequent flattening of the tail, which may give rise to the apparition of two tails, one on each side of the nucleus. It is also conceivable that other members of the solar system, when near enough, may produce similar, or partially similar, effects. The celebrated comet of 1744 exhibited "six tails spread out like an immense fan."



Comet of 1811.



Comet of 1744.

First Note on some Questions in Probability. *By the* Editor.

1. A correspondent, after remarking that Mr. Macdonald's attempt at a solution of Question 5863 (given on p. 54 of this Volume) seems to him utter nonsense, goes on to state that this question furnishes a

good example of the very subtle and obscure nature of many quesions on Probability. "Is the reasoning wrong," he adds, "or is it right; and consequently is the paradox to be accepted?" We have a quantity (B) which is certainly likely to be greater than £5, the odds being 4 : 3 in its favour (in fact, the *expectation*, for the cheque B, is greater than £5), and yet there is another unknown A, which is more likely to exceed B, than it is to exceed £5.

2. There is another question which, though of no unfrequent occurrence, and of great importance as lying at the root of a large number of questions in Probability, seems to require more investigation than it has hitherto received. If a bag contains a number of balls about which nothing is known, except that they are black or white, or—what is the same thing—if an event occurs a given number of times, about which we know nothing, except that it can only turn out in two ways, it is commonly assumed that every possible partition of the number into black and white is *à priori* equally likely. Now, if this is the case, say for two bags containing each 10 balls, and if they are all thrown into one bag, it is easy to show that it cannot be true for the 20 balls in this bag; yet how can our knowledge about these 20 balls be affected in any way by their having been originally in two sets? But, if the hypothesis be correct, it must be. Reducing the question to the simplest possible case, let two bags contain 1 ball each; if they are thrown together, the probabilities for the three possible partitions (0, 2), (1, 1), (2, 0) are as 1 : 2 : 1, instead of as 1 : 1 : 1, which the hypothesis gives.

SECOND NOTE ON SOME QUESTIONS IN PROBABILITY. *By the* EDITOR.

1. In reply to the EDITOR'S former *Note* hereon (see above) Mr. WHITWORTH makes the following observations:—

(*a*). "Professor CROFTON'S paradox (Question 5853) does not seem to me very paradoxical. It is easy to see, without calculation, that there is more chance of exceeding a fluctuating quantity than a constant quantity, even though the average value of the fluctuating quantity exceed the constant; for example, if the faces of a die be marked 11, 11, 11, 11, 11, 5, and the faces of another die be marked 7, 7, 7, 7, 7, 7, a person throwing with a *common* die has a chance ($= \frac{1}{36}$) of exceeding the first, and he has *no* chance of exceeding the second. In other words, his chance of exceeding 7 is less than his chance of exceeding a quantity which is "likely" to be 11, and which on an average is 10.

(*β*). "I cannot admit that, if a bag contains a number of balls, about which nothing is known except that they are black or white, *it is commonly assumed* that every possible partition of the *number* into black and white is *à priori* equally likely. What is (rightly) assumed is, that every possible partition of the *things* (treated as "different") is *à priori* equally likely. The chance that among the n balls there are exactly r black ones is therefore $\frac{n!}{r!\,n-r!}\left(\frac{1}{2}\right)^n$, and not $\frac{1}{n+1}$ (as the false assumption would give).* But, of course, the assumption which the EDITOR reprobates would be correct in such a case as the following:—A man is to paint n balls, some black and some white. He inquires how many are to be painted black, and the answer (r) is given at random. These balls are then put into a bag, &c."

---

* Mr. WOOLHOUSE states that, in his opinion, "this is undoubtedly the correct view."

2. On the foregoing observations, the before-cited correspondent makes the following remarks:—

(α). "No doubt Mr. WHITWORTH's explanation is the true one. The theory of probability abounds in paradoxes,—that is to say, cases where (to the rough common sense of mankind) the true view appears to be, *prima facie*, wrong. Thus, the drawer of the cheque from A, if given his choice to exchange it for £5, or for a cheque drawn from B, would, I think, generally prefer the £5, on being told that A was less likely to exceed £5 than to exceed B; yet in this he would be wrong.

(β). "There can be no doubt that *it is commonly assumed* that every possible partition of the *number* of balls into black and white is *à priori* equally likely, in a very important class of questions, including BAYES' problem; and, so far as I am aware, without any query or protest. For instance, such a question (TODHUNTER's *Algebra*, Art. 737) as: A bag contains 3 balls, and it is known that each ball is either black or white; a white ball has been drawn and replaced; what is the probability that another drawing will give a white ball? Here, on the above assumption, the answer is found to be $\frac{1}{2}$; whereas, had the antecedent probabilities of the 4 possible partitions been taken to be $\frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{1}{4}$, the answer would have been $\frac{3}{4}$,—which is, in fact, what we should have obtained by supposing that the fact of the drawn ball being white in no way affects the probabilities as to the 2 remaining balls. Questions referring to an event which can only turn out in one of 2 ways (see WILLIAMSON's *Integral Calculus*, p. 322; also TODHUNTER's *History*), are also always treated on the same hypothesis. The justice of that hypothesis certainly involves difficulties, as stated in the EDITOR's Note; that, however, which Mr. WHITWORTH approves of,—viz., that every partition of the *things* is equally likely,—involves consequences which shock common sense (at least, apparently) quite as much. It leads, in fact, to the conclusion that the observed cases do not affect the probability as to the remaining ones. I think it would be difficult to persuade any ordinary person, if told that a bag contains only black and white balls, and if he draws 2, which he finds to be white, that his next draw is as likely to be black as white; and much more, if he had drawn 3 or 4 or more white ones.

3. Mr. WHITWORTH, in answer to the foregoing paragraph (β), states that,—"Your correspondent would make out that, if a coin has been tossed several times, and has always fallen head, the odds are in favour of the next fall being tail;" and he adds that, "if this is so, there is no such thing as a mathematical theory of Probability." On this comment of Mr. WHITWORTH's, the said correspondent remarks as follows:—"The statement implies that a person who has drawn six balls in succession from a bag and found them all white, and who thinks the seventh the more likely to be white than black, is guilty of the same folly as one who, on tossing a shilling six times, and finding heads every time, concludes the seventh toss more likely to give head. In this idea (whether common sense be against him or not) he is at any rate at variance with all writers on the theory of Probability."

4. Mr. MACDONALD complains of what he calls the unmathematical term "nonsense," applied to his solution of Question 5853 by the distinguished contributor whose expression we reproduced in our first *Note* on Probabilities; and with a view to justify himself from the charge implied by the obnoxious epithet, he desires the correspondent in question to inform him to which of the statements made, (α), (β), (γ), (δ), &c., hereunder, the word "nonsense" is supposed more especially to apply.

In order nowise to misrepresent Mr. MACDONALD's defence, we give his remarks *verbatim*. They are as follows:—

"As my solution of Question 5853 has raised a discussion, perhaps you will allow me to make my view of it clearer. The paradox (if such there be) seems to me to lie in the word 'likely.' If you substitute for it its equivalent in the question, namely, 4 times out of 7, I can see nothing wonderful in the statement that 'there is a less chance that the cheque drawn from A exceeds £5 than that it exceeds an (unknown) quantity which is sometimes (*i.e.*, 4 out of 7) greater than £5.'

"I would put my solution thus:—

"(α). The chance that the cheque from A exceeds £5 is $\frac{3}{7}\frac{3}{6}$.

"(β). The chance that the cheque from A exceeds the one from B is $\frac{3}{7}\frac{3}{6}$.

"(γ). If the cheque from B is *always* greater than £5, the paradox is evident.

"(δ). But the cheque from B is *not always* greater than £5—only 4 times out of 7; hence my result.

"(ϵ). The chance that the cheque from A exceeds a quantity greater than £5 is only $\frac{5}{2}\frac{8}{4}\frac{8}{5}$."

5. On communicating Mr. MACDONALD's complaint to the contributor in question, he desires to apologise for the term "nonsense," which was only used by him to express familiarly the fact that he failed to take in Mr. MACDONALD's meaning—a failure which he is willing to admit to have been, probably, in part, his own fault. However, Mr. MACDONALD's solution seems to him, he adds, to miss the point intended by the question.

6. In regard to Art. 2 of the EDITOR's before-mentioned *Note*, Mr. MACDONALD sends the following remarks:—

"If the events are spoken of as merely so many black and so many white, it is clear that the number of such events will seem fewer when you consider 20 balls (say) in one set than when you take them in (say) two sets of 10 each. And for this reason: Suppose two bags, A and B, containing 10 each; then we may have 10 black in A, and 10 white in B, or 9 black and 1 white in A, and 1 black and 9 white in B, and so on. Now, if all 20 had been in one bag, each of these events would have been described as one, viz., 10 black and 10 white. The difficulty disappears if we consider not only the different events as above described, *but also the number of ways in which each event can happen*. To apply this to the example given in the *note*. If 2 balls are contained in a bag, then there are 4 possible arrangements, and not 3, *i.e.*, calling the balls P and Q, we may have (1), both black; (2), both white; (3), P black and Q white; and (4), P white and Q black—but as both (3) and (4) can be described as 1 black and 1 white, such an event has twice the probability of either 1 or 2."

5954. (By the Rev. Dr. HAUGHTON, F.R.S.)—The general and exact differential equations of motion of an ocean disturbed by any forces, and covering a solid nucleus, of any form, revolving on an axis, are, after LAPLACE, usually written thus:—

$$\frac{dp}{dr} = X\frac{x}{r} + Y\frac{y}{r} + Z\frac{z}{r} - \frac{1}{2r}\frac{d^2(r^2)}{dt^2} + \frac{1}{r}\left(\frac{dr}{dt}\right)^2 + r\left(\frac{d\theta}{dt}\right)^2 + r\sin^2\theta\left(n + \frac{d\phi}{dt}\right)^2 \quad \ldots\ldots (A),$$

$$\frac{dp}{d\theta} = -X(y^2+z^2)^{\frac{1}{2}} + \frac{Yxy+Zxz}{(y^2+z^2)^{\frac{1}{2}}} - \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + r^2\sin\theta\cos\theta\left(n+\frac{d\phi}{dt}\right)^2 \quad \ldots\ldots\ldots (B),$$

$$\frac{dp}{d\phi} = Zy - Yz - 2r\frac{dr}{dt}\sin^2\theta\left(n+\frac{d\phi}{dt}\right) - 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}\left(n+\frac{d\phi}{dt}\right) - r^2\sin^2\theta\frac{d^2\phi}{dt^2} \ldots\ldots\ldots (C);$$

p being the pressure at any point of the ocean; r, θ, ϕ the polar coordinates, viz., radius-vector, north polar distance, and longitude, of any point of the ocean; n the angular velocity of the rotating nucleus; x, y, z the coordinates of any point of the ocean referred to three rectangular axes fixed in space, x being the axis of rotation; and X, Y, Z the components of the disturbing forces, resolved parallel to the axes of x, y, z.

These important equations are usually deduced from the differential equations of motion, referred to fixed rectangular coordinate axes, by means of transformation of coordinates, by a process which will be admitted, by any one who has honestly gone through it, to be hideously repulsive.

I say, that the exact equivalents of equations (A), (B), (C) may be written down at sight, and without calculation, from elementary geometrical and mechanical principles. Required to show how this may be done.

I. *Solution by* Professor Townsend, M.A., F.R.S.

Denoting by ρ the projection of r on the plane of yz, so that

$$\rho^2 = y^2+z^2 = r^2\sin^2\theta,$$

and supposing, as is done in the equations, the density of the water to be unity, we have, by the general equations of fluid motion,

$$\frac{dp}{dx} = X - \frac{d^2x}{dt^2}, \quad \frac{dp}{dy} = Y - \frac{d^2y}{dt^2}, \quad \frac{dp}{dz} = Z - \frac{d^2z}{dt^2},$$

from which, by resolution along the three rectangular directions of r, θ, and ϕ, we get at once

$$\frac{dp}{dr} = \frac{1}{r}[Xx+Yy+Zz] - \frac{1}{r}\left[x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right]$$

$$= \frac{1}{r}[Xx+Yy+Zz] - \frac{1}{r}\left[r\frac{d^2r}{dt^2}+\frac{dr^2}{dt^2}\right] + \frac{1}{r}\left[\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right]$$

$$= \frac{1}{r}[Xx+Yy+Zz] - \frac{1}{2r}\left[\frac{d^2(r^2)}{dt^2}\right] + \frac{1}{r}\left[\frac{dr^2}{dt^2}+r^2\frac{d\theta^2}{dt^2}+\rho^2\left(n+\frac{d\phi}{dt}\right)^2\right]$$

$$\frac{dp}{d\theta} = -X\rho + [Yy+Zz]\frac{x}{\rho} + \rho\frac{d^2x}{dt^2} - \frac{x}{\rho}\left[y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right]$$

$$= -X\rho + [Yy+Zz]\frac{x}{\rho} + \rho\frac{d^2x}{dt^2} - x\frac{d^2\rho}{dt^2} + \rho x\left[n+\frac{d\phi}{dt}\right]^2$$

$$= -X\rho + [Yy+Zz]\frac{x}{\rho} - \frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] + \rho x\left[n+\frac{d\phi}{dt}\right]^2,$$

$$\frac{dp}{d\phi} = [Zy-Yz] - \left[y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right] = [Zy-Yz] - \frac{d}{dt}\left[\rho^2\left(n+\frac{d\phi}{dt}\right)\right]$$

$$= [Zy-Yz] - \frac{d}{dt}\left[r^2\sin^2\theta\left(n+\frac{d\phi}{dt}\right)\right],$$

which are manifestly the equivalents to the equations in question.

II. *Solution by* Professor MINCHIN, M.A.

Dr. HAUGHTON's equations can be at once written down if we consider the equilibrium of the usual polar element of mass under the action of the external forces and its own forces of inertia reversed. The areas of the three contiguous rectangular faces of this element are $r^2 \sin\theta\, d\theta\, d\phi$, $r \sin\theta\, dr\, d\phi$, and $r\, dr\, d\theta$, the normals to these being the radius vector, the meridian, and the parallel of latitude, respectively. Let P, T, S be the components of external force in these directions, and let $\ddot{r}_1$, $\ddot{\theta}_1$, $\ddot{\phi}_1$ be the accelerations of the element in the same directions.

Then denoting by p the *intensity* of pressure in the neighbourhood of the element, the pressure on the first face is $pr^2 \sin\theta\, d\theta\, d\phi$, and on the opposite face it is $\left(p + \frac{dp}{dr}\, dr\right) r^2 \sin\theta\, d\theta\, d\phi$; hence the excess of pressure along the radius vector is $\frac{dp}{dr}\, dv$, where dv is the volume of the element. Taking the excesses of pressure on the other faces in the same way, our equations of equilibrium are obviously

$$\frac{dp}{dr} = \rho\,(\mathrm{P} - \ddot{r}_1), \quad \frac{dp}{d\theta} = \rho r\,(\mathrm{T} - \ddot{\theta}_1), \quad \frac{dp}{d\phi} = \rho r \sin\theta\,(\mathrm{S} - \ddot{\phi}_1),$$

ρ denoting the density. These are the required equations at once, since the accelerations resulting from motions in radius vector, longitude, and latitude are, "at sight and without calculation,"

$$\ddot{r}_1 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta \,.\, \omega^2,$$

$$\ddot{\theta}_1 = \frac{1}{r}\,\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta \,.\, \omega^2,$$

$$\ddot{\phi}_1 = \frac{1}{r\sin\theta}\,\frac{d}{dt}\,(r^2\sin^2\theta \,.\, \omega),$$

where ω stands for the angular velocity in longitude and is obviously equal to $n + \frac{d\phi}{dt}$.

III. *Solution by* J. W. SHARPE, M.A.; J. E. A. STEGGALL, M.A.; *and others.*

Consider the portion of fluid contained within the usual polar element of space: let r, θ, ϕ be the coordinates of the point P to which this element becomes ultimately reduced. Draw OQ at right angles to Ox, OP; so that OQ lies in the plane yOz, and is at right angles to the plane through OP and Ox. To obtain the equations (A), (B), (C), we may consider the fluid element to be solidified. Let the density of the fluid be unity, as it is assumed to be in the given equations.

(i.) The form equation (A), calculate the acceleration of the fluid element in the direction of the radius vector OP.

The acceleration of a point fixed on the radius vector and momentarily coinciding with P is, in the direction of the radius vector,

$$-r\dot{\theta}^2 - r\sin^2\theta\,(n + \dot{\phi})^2;$$

where $\dot{\theta}$, $\ddot{\theta}$, &c., stand for $\frac{d\theta}{dt}$, $\frac{d^2\theta}{dt^2}$, &c.; for the motion of the fixed point is compounded of a motion of rotation about Ox, giving the acceleration

$$\frac{dw}{dt} = -Z + \frac{1}{\rho}\frac{dp}{dz} + \frac{Yx - Xy}{x^2 + y^2} - \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) + r^2\sin\theta\cos\theta\left(n + \frac{d\phi}{dt}\right)^2 \quad \text{..... (B),}$$

$$\frac{dv}{dt} = Y + \frac{1}{\rho}\frac{dp}{dy} - \frac{v}{r}\frac{dr}{dt} + \frac{w}{r}\left(n + \frac{d\phi}{dt}\right) - 2r\cos\theta\frac{d\theta}{dt}\left(n + \frac{d\phi}{dt}\right) - r\sin^2\theta\frac{d^2\phi}{dt^2} \quad \text{..... (C);}$$

p being the pressure at any point of the ocean; r, θ, ϕ the polar coordinates, viz. radius-vector, north polar distance, and longitude, of any point in the ocean; n the angular velocity of the rotating nucleus; x, y, z the coordinates of any point of the ocean referred to three rectangular axes fixed in space, z being the axis of rotation; and X, Y, Z the components of the disturbing force, parallel to the axes of x, y, z.

These are exact equations as are usually deduced from the differential equations of motion, referred to fixed rectangular coordinate axes, by means of transformation of the coordinates, by a process which will be admitted, by any one who has himself gone through it, to be hideously repulsive.

I ask that the exact equivalents of equations A, B, (C) may be written down at sight, and with no calculation from elementary geometrical and mechanical principles. I request to show how this may be done.

[illegible] Professor [illegible], M.A., F.R.S.

Denoting by s the projection of r on the plane of xy, so that

$$s^2 = x^2 + y^2 = r^2\sin^2\theta,$$

and supposing, as is usual in the equations, the density of the water to be unity, we have, by the general equations of fluid motion,

$$\frac{du}{dt} = X - \frac{d^2x}{dt^2} \qquad \frac{dv}{dt} = Y - \frac{d^2y}{dt^2} \qquad \frac{dw}{dt} = Z - \frac{d^2z}{dt^2}$$

from which, by resolution along the three rectangular directions of r, θ, and ϕ, we get at once

$$\frac{dw}{dt} = \frac{1}{\rho}\frac{dp}{dz} - Z = -\frac{1}{\rho}\left[x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right]$$

$$= \frac{1}{\rho}\frac{dp}{dz} - Z = -\frac{1}{\rho}\left[\frac{x}{r}\frac{d^2x}{dt^2} + \frac{y}{r}\frac{d^2y}{dt^2} + \frac{z}{r}\frac{d^2z}{dt^2}\right] - \frac{1}{r}\left[\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right]$$

$$= \frac{1}{\rho}\frac{dp}{dz} - Z = -\frac{1}{\rho}\frac{d^2r}{dt^2} - \frac{1}{r}\frac{dr^2}{dt^2} + r^2\frac{d\theta^2}{dt^2} + r^2\left(n + \frac{d\phi}{dt}\right)^2\Big]$$

$$\frac{dv}{dt} = -Y - \frac{1}{\rho}\frac{dp}{dy} = -Y - \frac{1}{s}\frac{dp}{dz}\frac{dz}{dy} = -Y - \frac{y}{s}\frac{d^2z}{dt^2} = -Y - \frac{y}{s}\left[y\frac{d^2y}{dt^2} + z\frac{d^2z}{dt^2}\right]$$

$$= -Y - \frac{1}{\rho}\frac{dp}{dy} = -Y - \frac{y}{s}\frac{d^2z}{dt^2} = -Y - \frac{y}{s}\frac{d^2y}{dt^2} + ys\left[n + \frac{d\phi}{dt}\right]^2$$

$$= -Y + [Y + Z]\frac{y}{s} - \frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] + ys\left[n + \frac{d\phi}{dt}\right]^2,$$

$$\frac{dv}{dt} = [Y - Y] - \frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] + [Y - Y] - \frac{d}{dt}\left[r^2\left(n + \frac{d\phi}{dt}\right)\right]$$

$$= [Y - \quad\quad \left(n + \frac{d\phi}{dt}\right)^2$$

which are [illegible] ions in question.

II. *Solution by* Professor MINCHIN, M.A.

Dr. HAUGHTON's equations can be at once written down if we consider the equilibrium of the usual polar element of mass under the action of the external forces and its own forces of inertia reversed. The areas of the three contiguous rectangular faces of this element are $r^2 \sin\theta\, d\theta\, d\phi$, $r \sin\theta\, dr\, d\phi$, and $r\, dr\, d\theta$, the normals to these being the radius vector, the meridian, and the parallel of latitude, respectively. Let P, T, S be the components of external force in these directions, and let $\ddot{r}_1, \ddot{\theta}_1, \ddot{\phi}_1$ be the accelerations of the element in the same directions.

Then denoting by p the *intensity* of pressure in the neighbourhood of the element, the pressure on the first face is $pr^2 \sin\theta\, d\theta\, d\phi$, and on the opposite face it is $\left(p + \frac{dp}{dr}\, dr\right) r^2 \sin\theta\, d\theta\, d\phi$; hence the excess of pressure along the radius vector is $\frac{dp}{dr}\, dv$, where dv is the volume of the element. Taking the excesses of pressure on the other faces in the same way, our equations of equilibrium are obviously

$$\frac{dp}{dr} = \rho(\mathrm{P} - \ddot{r}_1), \quad \frac{dp}{d\theta} = \rho r(\mathrm{T} - \ddot{\theta}_1), \quad \frac{dp}{d\phi} = \rho r \sin\theta(\mathrm{S} - \ddot{\phi}_1),$$

ρ denoting the density. These are the required equations at once, since the accelerations resulting from motions in radius vector, longitude, and latitude are, "at sight and without calculation,"

$$\ddot{r}_1 = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta \cdot \omega^2,$$

$$\ddot{\theta}_1 = \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) - r\sin\theta\cos\theta \cdot \omega^2,$$

$$\ddot{\phi}_1 = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2 \sin^2\theta \cdot \omega\right),$$

where ω stands for the angular velocity in longitude and is obviously equal to $n + \frac{d\phi}{dt}$.

III. *Solution by* J. W. SHARPE, M.A.; J. E. A. STEGGALL, M.A.; *and others.*

Consider the portion of fluid contained within the usual polar element of space: let r, θ, ϕ be the coordinates of the point P to which this element becomes ultimately reduced. Draw OQ at right angles to Oz, OP; so that OQ lies in the plane pOz, and is at right angles to the plane through OP and Oz. To [illegible] coordinates A, B, C, we may consider the fluid element to be situated. Let the density of the fluid be unity, as it is assumed to be in the [illegible].

(i.) The form equation (A), [illegible] the acceleration of the fluid element in the direction of the radius vector (r).

The acceleration of a point fixed in the radius vector and momentarily coinciding with P is, in the direction of the radius vector,

$$-r\dot{\theta}^2 - r\sin^2\theta\,\dot{\omega}\dot{\phi};$$

where $\dot{\theta}$, $\ddot{\theta}$, &c., stand for $\frac{d\theta}{dt}$, $\frac{d^2\theta}{dt^2}$, &c.; for the motion of the fixed point is compounded of a motion of [illegible] round Oz, [illegible] the acceleration

$r \sin\theta \,.\, (\dot{\phi}+n)^2$ directed towards that axis, together with a motion about OQ as axis giving an acceleration $r\dot{\theta}^2$ directed towards the origin.

The acceleration of the fluid element relatively to this point is, in the same direction, $\ddot{r}$; so that the total acceleration is

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,(n+\dot{\phi})^2.$$

The resolved part of the disturbing forces is

$$\left(\mathrm{X}\frac{x}{r}+\mathrm{Y}\frac{y}{r}+\mathrm{Z}\frac{z}{r}\right)dv,$$

where dv is the volume of the polar element, and therefore denotes the mass of the fluid contained in the element.

The fluid pressures which take effect in this direction are

$p \times$ (area of that face of the element which passes through P and is at right angles to OP)

$$-\left(p+\frac{dp}{dr}dr\right)\times(\text{area of the opposite face}) = -\frac{dp}{dr}dv;$$

hence, by D'Alembert's principle, reversing the effective forces, and writing down the equation of motion as one of equilibrium; or, by simply equating the motion produced and the forces producing it, we obtain

$$\frac{dp}{dr} = \mathrm{X}\frac{x}{r}+\mathrm{Y}\frac{y}{r}+\mathrm{Z}\frac{z}{r}-\ddot{r}+r\dot{\theta}^2+r\sin^2\theta\,(n+\dot{\phi})^2,$$

the equivalent of (A).

(ii.) To obtain equation (B), consider the moment of momentum about the line OQ, considering OQ to be a line fixed in space with which the line at right angles to the plane xOP momentarily coincides. Let ρ denote the projection of OP upon the plane yOz, so that $\rho^2 = y^2+z^2$.

The moment of momentum in the plane xOP is $r^2\dot{\theta}\,.\,dv$; its rate of change is $\frac{d}{dt}(r^2\dot{\theta})\times dv$; considering as positive that motion which tends to increase the angle θ. The acceleration in the plane at right angles to Ox is $-r\sin\theta\,(n+\dot{\phi})^2$, giving a moment of momentum

$$-r\sin\theta\,(n+\dot{\phi})^2\times r\cos\theta\times dv = -r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2\times dv;$$

hence moment of effective forces is

$$\left\{\frac{d}{dt}(r^2\dot{\theta})-r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2\right\}dv;$$

moment of disturbing forces is

$$\left\{-\mathrm{X}\rho+\mathrm{Y}\frac{y}{\rho}\times x+\mathrm{Z}\frac{z}{\rho}\times x\right\}dv;$$

moment of fluid pressures is

$$-\frac{dp}{d\theta}\times r\sin\theta\,d\phi\,.\,dr\times r = -\frac{dp}{d\theta}dv:$$

hence

$$\frac{dp}{d} = -\mathrm{X}(y^2+z^2)^{\frac{1}{2}}+\frac{\mathrm{Y}xy+\mathrm{Z}xz}{(y^2+z^2)^{\frac{1}{2}}}-\frac{d}{dt}(r^2\dot{\theta})+r^2\sin\theta\cos\theta\,(n+\dot{\phi})^2 \ldots \text{(B)}.$$

(iii.) For equation (C) takes moments in the same way about the axis of rotation Ox. The moment of momentum of the effective forces is clearly

$$r \sin\theta\,(n+\dot{\phi}) \times r \sin\theta \times dv,$$

of which the rate of change is

$$\frac{d}{dt}\left\{r^2 \sin^2\theta\,(n+\dot{\phi})\right\} \times dv.$$

The moment of the disturbing forces is $(Zy - Yz)\,dv$, and that of the fluid pressures is ultimately

$$-\frac{dp}{d\phi}\,d\phi \times dr \,.\, r\,d\theta \,.\, r \sin\theta = -\frac{dp}{d\phi}\,dv;$$

hence $\frac{dp}{d\phi} = Zy - Yz - \frac{d}{dt}\left\{r^2 \sin^2\theta\,(n+\dot{\phi})\right\}$, the equivalent of (C).

This method of obtaining the equations appears to satisfy Dr. HAUGHTON'S requirements, being perfectly direct, and having no reference, as in the case of Prof. TOWNSEND'S method, to transformation of coordinates. Each term in the equations as given above can be written down from first principles, without the necessity of writing down any preliminary calculations.

IV. *Solution by* Professor G. JOHNSTONE ALLMAN, M.A.

The following solution is extracted from the *Report* made by a Mathematical Committee, on January 30th, 1847, on a paper entitled "On the Polar Equations of Dynamics and Hydrostatics," and published in Vol. III. of the *Transactions* of the Dublin University Philosophical Society for the 4th and 5th sessions, ending November, 1847.

"Mr. ALLMAN commenced by stating a principle of considerable importance in many physical investigations, viz.,

The effective moment round any axis

$$= \frac{d}{dt}\left\{\text{Areolar velocity round that axis}\right\};$$

and thought it worth noticing that the truth of this principle may be established by a method similar to that used by Newton in his first Proposition. In the first place, let the motion of a single body, submitted to the action of given forces and moving freely, be considered.

Let the components of the given forces parallel to three fixed axes be X, Y, Z; let r be the radius vector of the centre of gravity of the body, ψ the angle between the radius and the plane of XY, and ϕ the angle which the plane of the radius vector and the axis of Z makes with the plane of ZX.

About the origin as centre and with the radius r let a sphere be described; let the motion of the body be resolved into three rectangular directions, one in the radius r, a second in the meridian plane Zr, and a third in the plane of the parallel circle; then the centrifugal forces, $\left(r\frac{d\psi^2}{dt^2}\right.$ acting out from the centre of the sphere, and $\left.r\cos\psi\,\frac{d\phi^2}{dt^2}\right)$ acting out from the centre of the parallel), arising from the inertia of the moving body, must be introduced in order to reduce the question to Statics.

Since the body moves freely, the effective force in the radius vector is equal to the sum of the applied forces in that direction, or,

$$(\mathrm{X}\cos\phi+\mathrm{Y}\sin\phi)\cos\psi+\mathrm{Z}\sin\psi+r\frac{d\psi^2}{dt^2}+r\cos^2\psi\frac{d\phi^2}{dt^2}-\frac{d^2r}{dt^2}=0.$$

Also, the effective moment round the axis of Z is equal to the sum of the applied moments round that axis, or

$$(\mathrm{Y}\cos\phi-\mathrm{X}\sin\phi)\,r\cos\psi-\frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\}=0.$$

Again, the effective moment round an axis perpendicular to the plane of the meridian is equal to the sum of the applied moments round that axis, or

$$\mathrm{Z}r\cos\psi-(\mathrm{X}\cos\phi+\mathrm{Y}\sin\phi)\,r\sin\psi-r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2}-\frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\}=0.$$

Mr. Allman next considered the case of a system of bodies regarded as material particles, mutually attracting each other, submitted to the action of forces directed to fixed centres and moving freely; and proceeded to form the general equation arising from the joint application of D'Alembert's principle, and the principle of virtual velocities, where the polar coordinates r, ϕ, ψ, r', ϕ', ψ', &c. of the different bodies m, m', &c., are taken as the variables.

By means of an important observation of Lagrange, *Mécanique Analytique*, Tome I., page 51, that 'if in the general formula of virtual velocities one of the independent variables be the angle of rotation around any axis, then the coefficient of the differential of this angle will be the sum of all the moments relative to this axis,' the general equation

$$\begin{aligned}
&\mathrm{S}m\left[(\mathrm{X}\cos\phi+\mathrm{Y}\sin\phi)\cos\psi+\mathrm{Z}\sin\psi+r\frac{d\psi^2}{dt^2}+r\cos^2\psi\frac{d\phi^2}{dt^2}-\frac{d^2r}{dt^2}\right]\delta r\\
&+\mathrm{S}m\left[(\mathrm{Y}\cos\phi-\mathrm{X}\sin\phi)\,r\cos\psi-\frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\}\right]\delta\phi\\
&+\mathrm{S}m\Big[\mathrm{Z}r\cos\psi-(\mathrm{X}\cos\phi+\mathrm{Y}\sin\phi)\,r\sin\psi\\
&\qquad-r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2}-\frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\}\Big]\delta\psi=0*
\end{aligned}$$

is established in a manner as direct and immediate as the equation

$$\mathrm{S}m\left\{\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\delta x+\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\delta y+\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\delta z\right\}=0.$$

In the case of the motion of fluids, the difference between the sum of the applied forces and the effective forces (also the difference between the sum of the applied moments and the effective moment) must equilibrate by means of the pressure; now the force in the radius vector arising from the pressure is $\frac{dp}{dr}$, the moment of the pressure round the axis of Z is

$$r\cos\psi\frac{dp}{r\cos\psi\,d\phi}=\frac{dp}{d\phi},$$

---

* This equation is deduced by Lagrange from the principle of least action, and is also an elegant consequence of the transformation of the general formula of dynamics given by him in the fourth section of the second part of the *Mécanique*.

and the moment of the pressure round an axis perpendicular to the plane of the meridian is $r\frac{dp}{rd\psi} = \frac{dp}{d\psi}$.

The polar equations of hydro-dynamics are therefore

$$(\mathrm{X}\cos\phi + \mathrm{Y}\sin\phi)\cos\psi + \mathrm{Z}\sin\psi + \frac{d\psi^2}{dt^2} + r\cos^2\psi\frac{d\phi^2}{dt^2} - \frac{d^2r}{dt^2} = \frac{dp}{dr},$$

$$(\mathrm{Y}\cos\phi - \mathrm{X}\sin\phi)\, r\cos\psi - \frac{d}{dt}\left\{r^2\cos^2\psi\frac{d\phi}{dt}\right\} = \frac{dp}{d\phi},$$

$$\mathrm{Z}r\cos\psi - (\mathrm{X}\cos\phi + \mathrm{Y}\sin\phi)\, r\sin\psi - r^2\sin\psi\cos\psi\frac{d\phi^2}{dt^2} - \frac{d}{dt}\left\{r^2\frac{d\psi}{dt}\right\} = \frac{dp}{d\psi}.$$

These are substantially the same as those given by Mr. AIRY in his article on *Tides and Waves*, in the *Encyclopedia Metropolitana*, p. 264."

V. *Solution by the* PROPOSER.

The complete differential equations of the motion of the sea or atmosphere, referred to polar coordinates, are regarded, justly, as one of the most brilliant results that we owe to the genius of Laplace; and yet they are found to be a "stumbling-block" in the way of young mathematicians, from the hideously repulsive form in which they are deduced, by transformation, from fixed rectilinear coordinates, by Laplace himself, and by his followers. Any attempt, therefore, to write down these equations at sight, from elementary geometrical and mechanical principles, will be regarded as useful. According to the self-evident principle of D'Alembert, all problems of Dynamics are reducible to problems of Statics, by introducing velocities and accelerating forces, equal and opposite to the existing velocities and accelerating forces. Now, the most general equations of equilibrium, of any system, are the following, six in number:—

$$\mathrm{X} = 0,\quad \mathrm{Y} = 0,\quad \mathrm{Z} = 0,\quad \mathrm{L} = 0,\quad \mathrm{M} = 0,\quad \mathrm{N} = 0 \ldots\ldots (1),$$

where X, Y, Z are the sums of the external forces resolved along three rectangular axes; and L, M, N are the sums of the couples (or twists) of those forces round the axes of X, Y, Z respectively.

The corresponding dynamical equations are these:—

$$\mathrm{X} - \frac{d^2x}{dt^2} = 0,\quad \mathrm{Y} - \frac{d^2y}{dt^2} = 0,\quad \mathrm{Z} - \frac{d^2z}{dt^2} = 0,\quad \mathrm{L} - \frac{d}{dt}\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right) = 0,$$

$$\mathrm{M} - \frac{d}{dt}\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right) = 0,\quad \mathrm{N} - \frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) = 0 \ldots\ldots (2).$$

In the case of an incompressible fluid these equations become

$$\frac{dp}{dx} = \mathrm{X} - \frac{d^2x}{dt^2},\quad \frac{dp}{dy} = \mathrm{Y} - \frac{d^2y}{dt^2},\quad \frac{dp}{dz} = \mathrm{Z} - \frac{d^2z}{dt^2} \ldots\ldots (3),$$

where p is the common pressure of the fluid (equal in all directions) at any point (x, y, z). It will be noted, that the last three equations of (2), depending on couples, disappear; because, in consequence of the mobility of the particles of the fluid *inter se*, internal couples, or twists, become impossible.

The Laplacian equations of motion, in polar coordinates, are usually deduced from (3), by transformation of the coordinates, from x, y, z, re-

ferred to fixed axes, where the axis of x is the axis of rotation; that of y an axis perpendicular to x, and fixed in space; and that of z, an axis perpendicular to those of x, y; to r, θ', ϕ', where r is the radius vector, θ' is the north polar distance, and ϕ' is the angular distance from the plane of x, y, of the meridian of any moving particle.

Instead of referring the forces to fixed coordinates, I refer them to the following movable rectangular coordinates:—

Axis of x'.—Let R denote the sum of the forces at any point, acting along the radius vector (*negative* towards the centre, and *positive* from it).

Axis of y'.—Let S denote the sum of the forces at any point acting in the meridional moving plane, and perpendicular to r (*positive* towards the equator, and *negative* towards the pole).

Axis of z'.—Let T denote the sum of the forces at any point acting perpendicularly to the two former directions, or in the direction of the tangent to the small circle of latitude (*negative* against the rotation, and *positive* with it. Let r, θ', ϕ' denote the polar coordinates in their most general form. The alteration in pressure produced by a change in r is similar to that produced by a change in x, y, z of the first three of equations (2) (because they are all linear magnitudes), and denotes a *force* acting to or from the centre; but the alteration in pressure produced by a change in angular direction by a change in θ' or ϕ' is no longer a *force*, but a *couple*, tending to turn the fluid round the centre. Thus,

$\frac{dp}{dr}$ is a *force* acting in the direction of the radius vector;

$\frac{dp}{d\theta'}$ is a *couple*, acting always in the moving meridional plane, and whose axis moves perpendicular to that plane;

$\frac{dp}{d\phi'}$ is a *couple*, acting always round the axis of rotation, and parallel to the equatorial plane.

It is evident that, if D'Alembert's equations (2) are satisfied—

(1) for *forces* acting along the radius vector;

(2) for *couples* acting in the meridional plane in every possible position of that plane;

(3) for couples acting always round the axis of rotation;

then complete dynamical equilibrium will be secured.

We may discount all the mechanical consequences of the rotation by introducing the centrifugal force, leaving only the geometrical consequences of the rotation, in the problem. The geometrical effect of the rotation is expressed by writing $\phi' = nt + \phi'$,

where n is the angular velocity of the earth's rotation.

The components of the velocity of any particle along R, S, T are

$$\frac{dr}{dt}, \quad r\frac{d\theta'}{dt}, \quad r\sin\theta'\left(n + \frac{d\phi'}{dt}\right).$$

The centrifugal force affects the directions R, S only, and does not enter into T. The centrifugal force in the direction of R is, obviously,

$$\frac{1}{r}\left\{\frac{r^2 d\theta'^2}{dt^2} + r^2\sin^2\theta'\left(n + \frac{d\phi'}{dt}\right)^2\right\}.$$

From this, and from the first three equations (2) we find, at sight,

$$\frac{dp}{dr} = \mathrm{R} - \frac{d^2r}{dt^2} + \frac{r d\theta'^2}{dt^2} - r\sin^2\theta'\left(n + \frac{d\phi'}{dt}\right)^2 \ldots\ldots\ldots (\mathrm{A}).$$

The centrifugal force in the direction of S is, obviously,

$$r \sin\theta' \cos\theta' \left(n + \frac{d\phi'}{dt}\right)^2.$$

The sixth of equations (2) therefore becomes, remembering that

$$y' \frac{dx'}{dt} - x' \frac{dy'}{dt} = r^2 \frac{d\theta'}{dt},$$

and, equating couples in the meridional plane,

$$\frac{dp}{d\theta'} = Sr - \frac{d}{dt}\left(r^2 \frac{d\theta'}{dt}\right) + r^2 \sin\theta' \cos\theta' \left(n + \frac{d\phi'}{dt}\right)^2 \quad \ldots\ldots \text{(B)}.$$

If we now equate the couples in the equatorial plane, we find, since

$$z \frac{dy}{dt} - y \frac{dz}{dt} = r^2 \sin^2\theta' \left(n + \frac{d\phi'}{dt}\right),$$

$$\frac{dp}{d\phi'} = Tr \sin\theta' - \frac{d}{dt}\left\{r^2 \sin^2\theta' \left(n + \frac{d\phi'}{dt}\right)\right\} \quad \ldots\ldots\ldots\ldots \text{(C)}.$$

The three equations just found from elementary principles are *exact equivalents* of the Laplacian differential equations, which are expressed by AIRY (in his *Tides and Waves*, p. 264) in the form given in the Question.

1401. (By the EDITOR).—Eliminate x from the equations

$$x^5 + px^2 + qx + r = 0, \quad x^4 + dx^3 + cx^2 + bx + a = y,$$

and exhibit the result in the form

$$y^5 + Ay^4 + By^3 + Cy^2 + Dy + E = 0,$$

giving the values of A, B, C, D, E in terms of p, q, r, a, b, c.

---

* *Solution by the late* Professor CLIFFORD, F.R.S.

If we multiply the second equation by x, and then each equation in succession by $x^4, x^3, \ldots, x^0$, we shall have *ten* simple equations from which to eliminate the *ten* quantities $x^9, x^8, x^7, \ldots, x^1, x^0$; and the result is (if we put a for $a - y$) the determinant

$$\begin{vmatrix}
1 & d & c & b & a & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & d & c & b & a & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & d & c & b & a & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & d & c & b & a & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & d & c & b & a & 0 \\
1 & 0 & 0 & p & q & r & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & p & q & r & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & p & q & r & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & p & q & r & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & p & q & r
\end{vmatrix} = 0.$$

This, however, I have not had the courage even to attempt to work out. It will evidently be divisible by r.

NOTE ON THE SOLUTION OF QUESTION 1401, *by the* EDITOR.

The foregoing solution is one of many early unpublished contributions sent to us by a mathematician whose recent loss to science our readers, in common with the whole scientific world, have especial reason to deplore. In now publishing, what the small space at our disposal when the solution was sent, *sixteen* years ago, did not then permit us to publish, we would join in a hope expressed by Mr. CLIFFORD in an accompanying letter, that some contributor familiar with the manipulation of determinants would develope the result in the form required by the question. The developed form may then be compared with the result obtained by Mr. HARLEY in his solution of the question, as given on pp. 38—40 of Vol. I. of our *Mathematical Reprints*. The question is connected with the problem of the reduction of the general quintic to a trinomial form, and has an especial reference to the transformation effected, so early as 1786, by ERLAND S. BRING, a Swedish mathematician. Reference may also be made to pages 8 and 67 of Vol. I. of the *Reprints*.

We may remark that the letters sent by Mr. CLIFFORD with his solutions were even more interesting and valuable than the solutions themselves, and that the very earliest of them, sent at a time when, in speaking of a kite-string problem, he remarks, in passing, that he had not yet left off flying kites, showed unmistakeable signs of the originality and power by which he was afterwards so pre-eminently distinguished. In a letter bearing date "Exeter, Sep. 10, 1863," Mr. CLIFFORD returns again to Question 1401, and states that, though he has several times tried to complete his solution, "the determinant has baffled me, from my not being accustomed to the manipulation of determinants; but I hope soon to be settled at Cambridge, and then I will try again."

Since the publication of Mr. CLIFFORD's Solution, with editorial note thereon, in the *Educational Times* for June 1879, several incomplete attempts at a development of the determinant have been sent in. Of these the only one that we can spare room for is contained in the following note by Mr. WOOLHOUSE:—

"I would remark that, in the foregoing Solution to Question 1401, the determinant stated is unnecessarily extended. The tenth simple equation being the only one that involves the tenth quantity x^0, both it and that quantity should evidently be suppressed, and the determinant, as indicating the elimination of nine quantities $x^9, x^8, \ldots x^1$, from nine simple equations, stated thus:—

$$\begin{vmatrix} 1 & d & c & b & a & 0 & 0 & 0 & 0 \\ 0 & 1 & d & c & b & a & 0 & 0 & 0 \\ 0 & 0 & 1 & d & c & b & a & 0 & 0 \\ 0 & 0 & 0 & 1 & d & c & b & a & 0 \\ 0 & 0 & 0 & 0 & 1 & d & c & b & a \\ 1 & 0 & 0 & p & q & r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & p & q & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & p & q & r & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & p & q & r \end{vmatrix} = 0.$$

The development of this determinant may be obtained by eliminating directly from the nine equations. The five quantities x^9, x^8, x^7, x^6, x^1 admit of very easy elimination, thereby reducing the work down to four equations amongst four quantities, that is, a determinant of the fourth order, the development of which consists of twenty-four products, written out according to the well known-formulæ."

Printed by C F. Hodgson and Son, 1, Gough Square, Fleet Street, London, E.C.